

# Four dimensional ambitwistor strings and form factors of local and Wilson line operators.

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## Abstract

We consider the description of form factors of local and Wilson line operators (reggeon amplitudes) in  $\mathcal{N} = 4$  SYM within the framework of four dimensional ambitwistor string theory. We present the explicit expressions for string composite operators corresponding to stress-tensor operator supermultiplet and Wilson line operator insertion. It is shown, that corresponding tree-level string correlation functions correctly reproduce previously obtained Grassmannian integral representations. As by product we derive four dimensional tree-level scattering equations representations for the mentioned form factors and formulate a simple gluing procedure used to relate operator form factors with on-shell amplitudes.

Keywords: ambitwistor strings, super Yang-Mills theory, off-shell amplitudes, form factors, Wilson lines, superspace, reggeons

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## 1 Introduction

Recently twistor string theories [1,2] played a crucial role in understanding and discovery of mathematical structures underlying scattering amplitudes in  $\mathcal{N} = 4$  super Yang-Mills and  $\mathcal{N} = 8$  supergravity in four dimensions. Based on Witten’s twistor string theory Roiban, Spradlin and Volovich (RSV) got the integral representation of  $\mathcal{N} = 4$  SYM tree level  $N^{k-2}$ MHV amplitudes as integrals over the moduli space of degree  $k - 1$  curves in super twistor space [3,4]. Further generalization of RSV result was performed by Cachazo, He and Yuan (CHY) via the introduction of so called *scattering equations* [5–9]. Within the latter  $\mathcal{N} = 4$  SYM amplitudes are expressed in terms of integrals over the marked points on the Riemann sphere, which are localized on the solutions of mentioned scattering equations. Next the CHY formulae were shown to come naturally from *ambitwistor string theory* [10,11], which was also used to obtain loop-level generalization of scattering equations representation [12–21].

Another close direction in the study of scattering amplitudes is related to their representation in terms of integrals over Grassmannians [22–27]. First, this representation allows natural unification of different BCFW [28,29] representations for tree level amplitudes and loop level integrands [22,23]. Second, it is ultimately related to the integrable structure behind  $\mathcal{N} = 4$  SYM S-matrix [30–34]. Moreover, the Grassmannian integral

representation also naturally relates perturbative  $\mathcal{N} = 4$  SYM and twistor string theories amplitudes [26]. Finally, the Grassmannian integral representation of scattering amplitudes has led to the discovery of geometrical structure of  $\mathcal{N} = 4$  SYM S-matrix (so called Amplituhedron) [35–42].

The purpose of this work is to derive scattering equations representation of form factors of local and Wilson line operators in  $\mathcal{N} = 4$  SYM from four dimensional ambitwistor string theory. Recently we have already provided such a derivation for the case of reggeon amplitude (Wilson line form factors) in [43]. This paper contains both extra details of the latter derivation together with its extension to the case of form factors of local operators. Up to the moment we already have scattering equations (connected prescription) representations for the form factors of operators from stress-tensor operator supermultiplet and scalar operators of the form  $\text{Tr}(\phi^m)$  [44, 45]. Also the connected prescription formulae were extended to Standard Model amplitudes [46]. Besides, there are several results for the Grassmannian integral representation of form factors of operators from stress-tensor operator supermultiplet [47–50] and Wilson line operator insertions [51, 52], see also [53] for a recent interesting duality for Wilson loop form factors. A very close subject is the twistor and Lorentz harmonic chiral superspace formulation of form factors and correlation functions developed in [54–60]. Finally, we should mention recent works on generalized unitarity, polytopes, Wilson loop and color-kinematics duality,  $Y$ -system and dilatation operator for form factors in  $\mathcal{N} = 4$  SYM, see [61–69] and references therein.

The form factors of local operators is quite developed topic in a literature. The latter may be viewed as amplitudes of the processes, in which classical field coupled through gauge invariant operator  $\mathcal{O}$  produces an on-shell quantum state. On the other hand form factors of Wilson line operators are less known. These objects should be more familiar to the reader as gauge invariant off-shell amplitudes [70–79] (also known as reggeon amplitudes in the framework of Lipatov’s effective lagrangian), which appear within the context of  $k_T$  or high-energy factorization [80–83] as well as in the study of processes at multi-regge kinematics. The general practice when studying form factors is to consider the case of local gauge invariant color singlet operators. However, one may also consider gauge invariant (the representation under global gauge transformation is not necessary singlet) non-local operators, for example Wilson loops (lines) or their products [51–53, 70–79]. An insertion of Wilson line operator will then correspond to the off-shell or reggeized gluon in such formulation.

This paper is organized as follows. After introducing necessary definitions for the form factors of operators from stress-tensor operator supermultiplet and Wilson line operator insertions in the next section 2 we proceed in section 3 with the recalling of four dimensional ambitwistor string theory formalism. In section 4 based on the details of the gluing procedure for form factors we present explicit expressions for string composite operators corresponding to stress-tensor operator supermultiplet and Wilson line operator insertions. Using new string composite operators we compute corresponding tree-level string correlation functions and show, that they correctly reproduce the results of Veronese map

of previously obtained Grassmannian integral representations. Section 5 contains the details of the mentioned Veronese map together with the discussion of relations between Grassmannian, link and scattering equations representations for the analysed form factors. In section 6 we use simple examples to show how the gluing procedure works at the level of particular amplitudes including the case with 3-point correlation function of reggeized gluons.

Finally, in section 7 we come with our conclusion and discuss possible future research directions. Appendices A and B contain the details of form factor gluing procedure for the cases of Grassmannian representation and loop integrands correspondingly.

## 2 Form factors of local and Wilson line operators

In this work we will be interested in ambitwistor string description of form factors of Wilson line and local operators in  $\mathcal{N} = 4$  SYM.  $\mathcal{N} = 4$  SYM is a maximally supersymmetric gauge theory in four space time dimensions [84, 85]. Its sixteen on-shell states (their creation/annihilation operators) could be conveniently described using  $\mathcal{N} = 4$  on-shell chiral superfield [86]:

$$\Omega = g^+ + \tilde{\eta}_A \psi^A + \frac{1}{2!} \tilde{\eta}_A \tilde{\eta}_B \phi^{AB} + \frac{1}{3!} \tilde{\eta}_A \tilde{\eta}_B \tilde{\eta}_C \epsilon^{ABCD} \bar{\psi}_D + \frac{1}{4!} \tilde{\eta}_A \tilde{\eta}_B \tilde{\eta}_C \tilde{\eta}_D \epsilon^{ABCD} g^-, \quad (2.1)$$

Here,  $g^+, g^-$  denote creation/annihilation operators of gluons with  $+1$  and  $-1$  helicities,  $\psi^A$  are creation/annihilation operators of four Weyl spinors with negative helicity  $-1/2$ ,  $\bar{\psi}_A$  are creation/annihilation operators of four Weyl spinors with positive helicity  $+1/2$  and  $\phi^{AB}$  stand for creation/annihilation operators of six scalars (anti-symmetric in the  $SU(4)_R$   $R$ -symmetry group indices  $AB$ ). All  $\mathcal{N} = 4$  SYM fields transform in the adjoint representation of  $SU(N_c)$  gauge group. In what follows we will also need superstates defined by the action of superfield creation/annihilation operators on vacuum. For  $n$ -particle superstate we have

$$|\Omega_1 \Omega_2 \dots \Omega_n\rangle \equiv \Omega_1 \Omega_2 \dots \Omega_n |0\rangle \quad (2.2)$$

Form factors of Wilson line operators are generally used to describe gauge invariant off-shell or reggeon amplitudes [70–79]. The Wilson line operators used to describe off-shell reggeized gluons are defined as [77]:

$$\mathcal{W}_p^c(k) = \int d^4x e^{ix \cdot k} \text{Tr} \left\{ \frac{1}{\pi g} t^c \mathcal{P} \exp \left[ \frac{ig}{\sqrt{2}} \int_{-\infty}^{\infty} ds p \cdot A_b(x + sp) t^b \right] \right\}. \quad (2.3)$$

where  $t^c$  is  $SU(N_c)$  generator<sup>1</sup> and we also used so called  $k_T$  - decomposition of the off-shell gluon momentum  $k$ ,  $k^2 \neq 0$ :

$$k^\mu = xp^\mu + k_T^\mu. \quad (2.4)$$

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<sup>1</sup>The color generators are normalized as  $\text{Tr}(t^a t^b) = \delta^{ab}$

Here,  $p$  is the off-shell gluon direction (also known as gluon polarization vector), such that  $p^2 = 0$ ,  $p \cdot k = 0$  and  $x \in [0, 1]$ . Such decomposition is generally parametrized by an auxiliary light-cone four-vector  $q^\mu$ , so that

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) = \frac{q \cdot k}{q \cdot p} \quad \text{and} \quad q^2 = 0. \quad (2.5)$$

As momentum  $k_T^\mu$  is transverse with respect to both  $p^\mu$  and  $q^\mu$  vectors one can decompose it into the basis of two “polarization” vectors<sup>2</sup> as [70]:

$$k_T^\mu(q) = -\frac{\kappa}{2} \frac{\langle p | \gamma^\mu | q \rangle}{[pq]} - \frac{\kappa^*}{2} \frac{\langle q | \gamma^\mu | p \rangle}{\langle qp \rangle} \quad \text{with} \quad \kappa = \frac{\langle q | \not{k} | p \rangle}{\langle qp \rangle}, \quad \kappa^* = \frac{\langle p | \not{k} | q \rangle}{[pq]}. \quad (2.6)$$

It is easy to see, that  $k^2 = -\kappa\kappa^*$  and both  $\kappa$  and  $\kappa^*$  are independent of auxiliary four-vector  $q^\mu$  [70]. Another useful relation, which is direct consequence of  $k_T$  decomposition and will be used often in practical calculations later on, is

$$k|p\rangle = |p\rangle\kappa^*. \quad (2.7)$$

Note, that Wilson line operator we use to describe off-shell gluon is colored. It is invariant  $\delta\mathcal{W}_p^c(k) = 0$  under local infinitesimal gauge transformations  $\delta A_\mu = [D_\mu, \chi]$  with  $\chi$  decreasing at  $x \rightarrow \infty$ . At the same time it transforms under global adjoint transformations of  $SU(N_c)$  with constant  $\chi$  as [73, 74]:

$$\delta\mathcal{W}_p(k) = g[\mathcal{W}_p(k), \chi]. \quad (2.8)$$

The form factor of Wilson line operator or gauge invariant amplitude with one off-shell and  $n$  on-shell gluons is then given by [77]:

$$\mathcal{A}_{n+1}(1^\pm, \dots, n^\pm, g_{n+1}^*) = \langle \{k_i, \epsilon_i, c_i\}_{i=1}^n | \mathcal{W}_p^{c_{n+1}}(k) | 0 \rangle, \quad (2.9)$$

where asterisk denotes off-shell gluon, while  $p, k, c$  stand for its direction, momentum and color index. Next  $\langle \{k_i, \epsilon_i, c_i\}_{i=1}^m | = \bigotimes_{i=1}^m \langle k_i, \epsilon_i, c_i |$  and  $\langle k_i, \epsilon_i, c_i |$  denotes on-shell gluon state with momentum  $k_i$ , polarization vector  $\epsilon_i^-$  or  $\epsilon_i^+$  and color index  $c_i$ . Also in the case when there is no confusion in the position of Wilson line operator insertion the latter will be labeled just by  $g^*$ . Form factors with multiple Wilson line insertions or amplitudes with multiple off-shell gluons can be represented in a similar fashion:

$$\mathcal{A}_{m+n}(1^\pm, \dots, m^\pm, g_{m+1}^*, \dots, g_{n+m}^*) = \langle \{k_i, \epsilon_i, c_i\}_{i=1}^m | \prod_{j=1}^n \mathcal{W}_{p_{j+m}}^{c_{j+m}}(k_{j+m}) | 0 \rangle, \quad (2.10)$$

where  $p_i$  is the direction of the  $i$ 'th ( $i = 1, \dots, n$ ) off-shell gluon and  $k_i$  is its off-shell momentum. As a function of kinematical variables this amplitude is written as

$$\mathcal{A}_{m+n}(1^\pm, \dots, g_{n+m}^*) = \mathcal{A}_{m+n}\left(\{\lambda_i, \tilde{\lambda}_i, \pm, c_i\}_{i=1}^m; \{k_j, \lambda_{p,j}, \tilde{\lambda}_{p,j}, c_j\}_{j=m+1}^{m+n}\right), \quad (2.11)$$

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<sup>2</sup>Here we used the helicity spinor decomposition of light-like four-vectors  $p$  and  $q$ .

where  $\lambda_{p,j}, \tilde{\lambda}_{p,j}$  are spinors coming from helicity spinor decomposition of polarization vector of  $j$ 'th reggeized gluon. In the case when only off-shell gluons are present (correlation function of Wilson line operator insertions) we have:

$$\mathcal{A}_{0+n}(g_1^* \dots g_n^*) = \langle 0 | \mathcal{W}_{p_1}^{c_1}(k_1) \dots \mathcal{W}_{p_n}^{c_n}(k_n) | 0 \rangle. \quad (2.12)$$

Of course it is also possible to consider color ordered versions of Wilson line form factors, while the original off-shell amplitudes (Wilson line form factors) are then recovered using color decomposition<sup>3</sup>:

$$\begin{aligned} \mathcal{A}_{n+m}^*(1^\pm, \dots, m^\pm, g_{m+1}^*, \dots, g_{n+m}^*) &= g^{n-2} \sum_{\sigma \in S_{n+m}/Z_{n+m}} \text{tr}(t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n+m)}}) \times \\ &\times A_{n+m}^*(\sigma(1^\pm), \dots, \sigma(g_{n+m}^*)). \end{aligned} \quad (2.13)$$

Note, that in the planar limit this decomposition is valid both for arbitrary tree and loop level amplitudes.

In the case of  $\mathcal{N} = 4$  SYM one may also consider other than gluons on-shell states from  $\mathcal{N} = 4$  supermultiplet. The corresponding  $\mathcal{N} = 4$  SYM superamplitudes are then given by

$$A_{m+n}^*(\Omega_1, \dots, \Omega_m, g_{m+1}^*, \dots, g_{n+m}^*) = \langle \Omega_1 \dots \Omega_m | \prod_{j=1}^n \mathcal{W}_{p_{m+j}}(k_{m+j}) | 0 \rangle, \quad (2.14)$$

and the explicit dependence of  $A_{m+n}^*(\Omega_1, \dots, g_{m+n}^*)$  amplitude on kinematical variables takes the form

$$A_{m+n}^*(\Omega_1, \dots, g_{m+n}^*) = A_{m+n}^* \left( \{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^m; \{k_j, \lambda_{p,j}, \tilde{\lambda}_{p,j}\}_{j=m+1}^{m+n} \right). \quad (2.15)$$

The above superamplitude contains not only component amplitudes with on-shell gluons, but also all amplitudes with other on-shell states from  $\mathcal{N} = 4$  supermultiplet. The helicity spinors  $\lambda_i, \tilde{\lambda}_i$  encode kinematics of on-shell states, while  $\tilde{\eta}_i$  encodes their helicity content. Off-shell momentum  $k_i$  and light-cone direction vector  $p_i = \lambda_{p,i} \tilde{\lambda}_{p,i}$  encode information related to Wilson line operator insertion. So, in what follows we will be considering partially supersymmetrized version of amplitudes (2.10) with on-shell states treated in supersymmetric manner, while Wilson line operators ("off-shell states") left unsupersymmetrized. The component amplitudes containing gluons, scalars and fermions may then be extracted as coefficients in  $\tilde{\eta}$  expansion of  $A_{m+n}^*$  superamplitude similar to the case of ordinary on-shell amplitudes and super form factors.

While our present consideration should be applicable<sup>4</sup> not only to Wilson line but to arbitrary local operators, here for concreteness we will restrict ourselves to the case

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<sup>3</sup>See for example [51, 87].

<sup>4</sup>See corresponding discussion in Conclusion.

of operators from stress-tensor operator supermultiplet. When considering the latter the general practice is to focus on the chiral part of this multiplet. Using harmonic superspace approach [88, 89] it is given by [89–92]:

$$\mathcal{T}(x, \theta^+) = \text{tr}(\phi^{++}\phi^{++}) + \dots + \frac{1}{3}(\theta^+)^4 \mathcal{L}, \quad (2.16)$$

where  $u_A^{+a}$ ,  $u_A^{-a'}$  is a set of harmonic coordinates parameterizing coset  $\frac{SU(4)}{SU(2) \times SU(2)' \times U(1)}$  and  $\theta_\alpha^{+a} = \theta_\alpha^A u_A^{+a}$ ,  $\theta_\alpha^{-a'} = \theta_\alpha^A u_A^{-a'}$ . Here,  $A$  is  $SU(4)_R$  index,  $a$  and  $a'$  are  $SU(2)$  indices and  $\pm$  denote  $U(1)$  charges. For example  $\epsilon^{ab}\phi^{++} = \phi^{AB}u_A^{+a}u_A^{+b}$ , where  $\phi^{AB}$  is the scalar field from  $\mathcal{N} = 4$  lagrangian. The color ordered form factors of operators from the chiral truncation of stress-tensor operator supermultiplet  $F_n$  are then given by

$$F_n(\Omega_1, \dots, \Omega_n; \mathcal{T}) \equiv \langle \Omega_1 \dots \Omega_n | \mathcal{T}(q, \gamma^-) | 0 \rangle = F_n \left( \{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^n; \{q, \gamma^-\} \right), \quad (2.17)$$

where  $\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^n$  are kinematical and helicity data of the on-shell states,  $q$  is the operator momentum and  $\gamma^-$  parametrizes the operator content of the chiral part of  $\mathcal{N} = 4$  SYM stress-tensor operator supermultiplet. Here, we have also performed the Fourier transformation from variables  $x, \theta^+$  to  $q, \gamma^-$  [90, 91]. The full physical form factor may then be restored from its color ordered version using standard color decomposition formula

$$\begin{aligned} \mathcal{F}_n(\Omega_1, \dots, \Omega_n; \mathcal{T}) &= g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{tr} (t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}}) \times \\ &\times F_n(\sigma(\Omega_1), \dots, \sigma(\Omega_n); \mathcal{T}), \end{aligned} \quad (2.18)$$

where  $S_n/Z_n$  denotes all none cyclic permutations of  $n$  objects. As in the case of of shell amplitudes this formula is valid both for arbitrary tree and loop level form factors in the planar limit<sup>5</sup>. At least at tree level the form factors of full stress-tensor operator supermultiplet can be reconstructed if the explicit form of (2.17) is known [90].

### 3 Four dimensional ambitwistor strings

As was already mentioned in Introduction to describe form factors of local and Wilson line operators we will be using the four dimensional ambitwistor string theory originally formulated in [11]. Our presentation of this theory here closely follows [11] and we refer the interested reader to this original paper and [93] for further details.

The target space of four dimensional ambitwistor string is given by projective ambitwistor space  $\mathbb{PA}$ . The latter is the supersymmetrized space of complex null geodesics

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<sup>5</sup>At loop level one should take into account appropriate powers of t'Hooft coupling constant  $g^2 N_c$ , which were suppressed here.

in a complexified Minkowski given by a quadric  $Z \cdot W = 0$  inside the product of twistor and dual twistor spaces  $\mathbb{PT} \times \mathbb{PT}^*$  quotient by relative scaling  $Z \cdot \partial_Z - W \cdot \partial_W$ :

$$\mathbb{PA} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* \mid Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}. \quad (3.19)$$

In the case of  $\mathcal{N}$  supersymmetries  $Z = (\lambda_\alpha, \mu^{\dot{\alpha}}, \chi^r) \in \mathbb{T} = \mathbb{C}^{4|\mathcal{N}}$ ,  $W = (\tilde{\mu}, \tilde{\lambda}, \tilde{\chi}) \in \mathbb{T}^*$  and  $Z \cdot W = \lambda_\alpha \tilde{\mu}^\alpha + \mu^{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} + \chi^r \tilde{\chi}_r$ , where  $\chi, \tilde{\chi}$  are fermionic,  $\alpha = 0, 1, \dot{\alpha} = \dot{0}, \dot{1}$  and  $r = 1, \dots, \mathcal{N}$  is R-symmetry index. The point  $(x, \theta, \bar{\theta})$  in non-chiral super Minkowski space corresponds to a quadric  $\mathbb{CP}^1 \times \mathbb{CP}^1$  parametrized by  $(\lambda, \tilde{\lambda})$  spinors. The correspondence is realized by the standard twistor incidence relations

$$\mu^{\dot{\alpha}} = i(x^{\alpha\dot{\alpha}} + i\theta^{r\alpha}\tilde{\theta}_r^{\dot{\alpha}})\lambda_\alpha, \quad \chi^r = \theta^{r\alpha}\lambda_\alpha, \quad (3.20)$$

$$\tilde{\mu}^\alpha = -i(x^{\alpha\dot{\alpha}} - i\theta^{r\alpha}\tilde{\theta}_r^{\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}}, \quad \tilde{\chi}_r = \tilde{\theta}_r^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}, \quad (3.21)$$

It is easy to check, that this quadric lies in  $Z \cdot W = 0$

The four dimensional ambitwistor string consists from worldsheet spinors  $(Z, W)$  with values in  $\mathbb{T} \times \mathbb{T}^*$  and  $\text{GL}(1, \mathbb{C})$  gauge field  $a$  acting as a Lagrange multiplier for the constraint  $Z \cdot W = 0$ . In the conformal gauge the action is given by<sup>6</sup>

$$S = \frac{1}{2\pi} \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W + a Z \cdot W + S_J, \quad (3.22)$$

where  $\bar{\partial} = d\bar{\sigma}\partial_{\bar{\sigma}}$  ( $\sigma, \bar{\sigma}$  are some local holomorphic and anti-holomorphic coordinates on Riemann surface  $\Sigma$ ) and  $S_J$  is the action for a worldsheet Kac-Moody current algebra  $J \in \Omega^0(\Sigma, K_\Sigma \otimes \mathfrak{g})$  for some Lie algebra  $\mathfrak{g}$ . Here  $K_\Sigma$  denotes canonical bundle on surface  $\Sigma$  and the remaining worldsheet fields take values in

$$Z \in \Omega^0(\Sigma, K_\Sigma^{1/2} \otimes \mathbb{T}), \quad (3.23)$$

$$W \in \Omega^0(\Sigma, K_\Sigma^{1/2} \otimes \mathbb{T}^*), \quad (3.24)$$

$$a \in \Omega^{0,1}(\Sigma), \quad (3.25)$$

where powers of canonical bundle denote fields conformal weights. The above action is invariant under a gauge symmetry

$$Z^I \rightarrow e^\gamma Z^I, \quad W_I \rightarrow e^{-\gamma} W_I, \quad a \rightarrow a - 2\bar{\partial}\gamma, \quad (3.26)$$

that quotients the target space by the action of  $Z \cdot \partial_Z - W \cdot \partial_W$ . The gauge fixing of worldsheet diffeomorphism symmetry<sup>7</sup> and the above gauge redundancy via standard

<sup>6</sup>It is obtained by chiral pullback of contact stucture on ambitwistor space  $\Theta = \frac{i}{2}(Z \cdot dW - W \cdot dZ)$  [11]. Note, that similar action first appeared in [2] in the context of open twistor string theory.

<sup>7</sup>In a general gauge, the  $\bar{\partial}$  operator in (3.22) is replaced by operator  $\bar{\partial}_{\tilde{e}} = \bar{\partial} + \tilde{e}\partial$  parametrizing the worldsheet diffeomorphism freedom.



BRST procedure leads to the introduction of the standard reparametrization (Virasoro)  $(b, c)$  together with  $GL(1)$   $(u, v)$  ghost systems:

$$c \in \Pi\Omega^0(\Sigma, T_\Sigma), \quad v \in \Pi\Omega^0(\Sigma), \quad (3.27)$$

$$b \in \Pi\Omega^0(\Sigma, K_\Sigma^2), \quad u \in \Pi\Omega^0(\Sigma, K_\Sigma), \quad (3.28)$$

where  $T_\Sigma$  denotes tangent bundle on surface  $\Sigma$  and  $\Pi\Omega^0(\Sigma, E)$  denotes the space of fermion-valued sections of  $E$ . The full worldsheet action is then given by

$$S = \frac{1}{2\pi} \int_\Sigma W \cdot \bar{\partial}Z - Z \cdot \bar{\partial}W + b\bar{\partial}c + u\bar{\partial}v + S_J, \quad (3.29)$$

and the BRST operator takes the form

$$Q = \oint cT + vZ \cdot W + Q_{\text{gh}}. \quad (3.30)$$

where  $T = W \cdot \partial Z - Z \cdot \partial W + T_J$  is the world-sheet stress-energy tensor.

## 4 String vertexes and correlation functions

To calculate string scattering amplitudes we need vertex operators. In general they are given by first-quantized wave functions of external states translated into worldsheet operator insertions. Penrose transform allows us to relate solutions to massless field equations in Minkowski space to cohomology classes on projective twistor space. In the case of Yang-Mills theory ambitwistor string vertex operators are obtained by pairing pullbacks of general ambitwistor space wave functions  $\alpha \in H^1(\mathbb{PA}, \mathcal{O})$  ( $\bar{\partial}$ -closed worldsheet  $(0, 1)$ -forms) with Kac-Moody currents  $J \cdot T_a$  to get  $\mathcal{V}_a = \int_\Sigma \alpha J \cdot T_a$ . For two types of momentum eigenstates (pullbacks from either twistor or dual twistor space) we get [11]:

$$\mathcal{V}_a = \int \frac{ds_a}{s_a} \bar{\delta}^2(\lambda_a - s_a \lambda) e^{is_a([\mu \bar{\lambda}_a] + \chi^r \bar{\eta}_{ar})} J \cdot T_a, \quad (4.31)$$

$$\tilde{\mathcal{V}}'_a = \int \frac{ds_a}{s_a} \bar{\delta}^2(\tilde{\lambda}_a - s_a \tilde{\lambda}) e^{is_a(\langle \tilde{\mu} \lambda_a \rangle + \tilde{\chi}_r \eta_a^r)} J \cdot T_a, \quad (4.32)$$

where  $\bar{\delta}(z) = \bar{\partial}(1/2\pi iz)$  for complex  $z$ . Note, that these vertex operators are  $Q$ -closed<sup>8</sup> and satisfy  $\{Q, \mathcal{V}_a\} = \{Q, \tilde{\mathcal{V}}'_a\} = 0$ . To facilitate further comparison with Grassmannian integral representation it is convenient to introduce slightly different representation for the second vertex operator. It is obtained by a Fourier transform<sup>9</sup> of the  $\eta$ 's into  $\tilde{\eta}$ 's:

$$\tilde{\mathcal{V}}_a = \int \frac{ds_a}{s_a} \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_a - s_a \tilde{\lambda} | \tilde{\eta}_a - s_a \tilde{\chi}) e^{is_a \langle \tilde{\mu} \lambda_a \rangle} J \cdot T_a. \quad (4.33)$$

<sup>8</sup>It should be noted, that in general this theory is anomalous and has nonzero central charge, so that  $Q^2 \neq 0$  [11, 93]

<sup>9</sup>Note, that in [11] instead a Fourier transform for the first operator from  $\eta$ 's to  $\tilde{\eta}$ 's was performed. The Grassmann part of the delta function is defined as usual  $\delta^{0|\mathcal{N}}(\tilde{\eta}) = \prod_{r=1}^{\mathcal{N}} \tilde{\eta}_r$ .

In the case of  $\mathcal{N} = 3$  these vertex operators together encode all sixteen degrees of freedom of  $\mathcal{N} = 4$  SYM theory. For  $\mathcal{N} = 4$  on the other hand each of them contains all  $\mathcal{N} = 4$  SYM on-shell states. In our consideration of  $\mathcal{N} = 4$  SYM to obtain maximally supersymmetric superamplitudes we will use the second option and use these vertex operators interchangeably.

$N^{k-2}$ MHV superamplitudes may be then obtained for example as correlation functions of  $k$  operators from dual twistor space and  $n - k$  operators from twistor space [11] (here and below we omit color structures and already work with color ordered objects):

$$A_{k,n} = \left\langle \tilde{\mathcal{V}}_1 \dots \tilde{\mathcal{V}}_k \mathcal{V}_{k+1} \dots \mathcal{V}_n \right\rangle. \quad (4.34)$$

Instead of computing the infinite number of contractions required by exponentials in vertex operators it is convenient to take exponentials into the action as sources

$$\int_{\Sigma} \sum_{i=1}^k i s_i \langle \tilde{\mu} \lambda_i \rangle \bar{\delta}(\sigma - \sigma_i) + \sum_{p=k+1}^n i s_p ([\mu \tilde{\lambda}_p] + \chi \tilde{\eta}_p) \bar{\delta}(\sigma - \sigma_p).$$

The corresponding equations of motion from this new action are then given by

$$\bar{\partial}_{\sigma} Z = \bar{\partial}(\lambda, \mu, \chi) = \sum_{i=1}^k s_i (\lambda_i, 0, 0) \bar{\delta}(\sigma - \sigma_i), \quad (4.35)$$

$$\bar{\partial}_{\sigma} W = \bar{\partial}(\tilde{\mu}, \tilde{\lambda}, \tilde{\chi}) = \sum_{p=k+1}^n s_p (0, \tilde{\lambda}_p, \tilde{\eta}_p) \bar{\delta}(\sigma - \sigma_p), \quad (4.36)$$

As  $(Z, W)$  fields are worldsheet spinors the solutions to the above equations are unique<sup>10</sup> and given by

$$Z(\sigma) = (\lambda, \mu, \chi) = \sum_{i=1}^k \frac{s_i (\lambda_i, 0, 0)}{\sigma - \sigma_i}, \quad (4.37)$$

$$W(\sigma) = (\tilde{\mu}, \tilde{\lambda}, \tilde{\chi}) = \sum_{p=k+1}^n \frac{s_p (0, \tilde{\lambda}_p, \tilde{\eta}_p)}{\sigma - \sigma_p}. \quad (4.38)$$

Then the path integrals over  $(Z, W)$  fields localize on the solutions (4.37)-(4.38), while current correlator contributes Parke-Taylor factor and for the color ordered on-shell amplitude we get [11]:

$$A_{n,k} = \int \frac{1}{\text{Vol GL}(2, \mathbb{C})} \prod_{a=1}^n \frac{ds_a d\sigma_a}{s_a (\sigma_a - \sigma_{a+1})} \prod_{p=k+1}^n \bar{\delta}^2(\lambda_p - s_p \lambda(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_i - s_i \tilde{\lambda}(\sigma_i), \tilde{\eta}_i - s_i \tilde{\chi}(\sigma_i)). \quad (4.39)$$

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<sup>10</sup>There no fermion zero modes on sphere.

Note, that ghosts  $c$  and  $v$  develop<sup>11</sup>  $n_c = 3$  (number of conformal Killing vectors on sphere) and  $n_v = 1$  zero modes correspondingly, which result in the  $GL(2, \mathbb{C})$  quotient above. In terms of homogeneous coordinates on Riemann sphere  $\sigma_\alpha = \frac{1}{s}(1, \sigma)$  the rescaled by a factor  $1/s$   $W$  and  $Z$  fields could be written as

$$Z(\sigma) = \sum_{i=1}^k \frac{(\lambda_i, 0, 0)}{(\sigma \sigma_i)}, \quad W(\sigma) = \sum_{p=k+1}^n \frac{(0, \tilde{\lambda}_p, \tilde{\eta}_p)}{(\sigma \sigma_p)}, \quad (4.40)$$

where  $(ij) = \sigma_{i\alpha} \sigma_j^\alpha$ . Then the final formula for the above amplitude takes the form [11]:

$$A_{n,k} = \int \frac{1}{\text{Vol } GL(2, \mathbb{C})} \prod_{a=1}^n \frac{d^2 \sigma_a}{(a \ a+1)} \prod_{p=k+1}^n \bar{\delta}^2(\lambda_p - \lambda(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i), \tilde{\eta}_i - \tilde{\chi}(\sigma_i)). \quad (4.41)$$

The scattering equations are then follow from the support of the delta functions

$$k_a \cdot P(\sigma_a) = \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}} P_{\alpha\dot{\alpha}}(\sigma_a) = \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}} \lambda_\alpha(\sigma_a) \tilde{\lambda}_{\dot{\alpha}}(\sigma_a) = 0. \quad (4.42)$$

It is important to note that the exact form of scattering equations themselves and scattering equations representations for amplitudes depends on which particle vertex operators were taken as  $\tilde{\mathcal{V}}_i$  and which as  $\mathcal{V}_i$  in (4.34). So, we have several equivalent representations for  $A_{k,n}$ . Their existence as we will see in section 5 is related to the  $GL(k)$  "gauge invariance" of Grassmannian integral representation for scattering amplitudes.

## 4.1 Wilson line operator insertion

The ambitwistor string vertex operators for either Wilson line or stress-tensor operator insertions could be also obtained from the pullbacks of corresponding ambitwistor space wave functions to string worksheet<sup>12</sup>. The required ambitwistor space wave functions could be easily found using a representation of corresponding off-shell amplitudes (form factors) with  $n+1$  legs in terms of convolutions of corresponding minimal off-shell amplitude (form factor) with on-shell amplitudes with  $n+2$  legs. The motivations and checks of these representations could be found in Appendix A for Grassmannian integral representation of on-shell amplitudes and in Section 6 for explicit amplitude expressions. Appendix B contains a simple check of similar representation for the integrands of loop level off-shell

<sup>11</sup>This is easy to see with the help of Riemann-Roch theorem recalling that  $\deg T_\Sigma = -\deg K_\Sigma = 2g-2$ , where  $g$  is the genus of Riemann surface.

<sup>12</sup>For the previous work on reggeon string vertexes within the context of superstring theory see [94, 95] and references therein.

amplitudes. This way the ambitwistor string vertex operator for Wilson line insertion could be written as

$$\mathcal{V}_{n,n+1}^{\text{WL}} = \int \prod_{i=n}^{n+1} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i}{\text{Vol}[\text{GL}(1)]} d^4 \tilde{\eta}_i A_{2,2+1}^*(g^*, \Omega_n, \Omega_{n+1}) \Big|_{\lambda \rightarrow -\lambda} \mathcal{V}_n \mathcal{V}_{n+1} \Big|_{T^a T^b \rightarrow i f^{abc} T^c \rightarrow T^c}, \quad (4.43)$$

where the vertex is supposed to be inserted at points  $\sigma_n, \sigma_{n+1}$ ,  $c$  is the color index of off-shell gluon and we have used projection of tensor product of two adjoint on-shell gluon color representations onto off-shell gluon adjoint color representation. The minimal off-shell amplitude  $A_{2,2+1}^*(g^*, \Omega_n, \Omega_{n+1})$  is given by [51]:

$$\begin{aligned} A_{2,2+1}^*(g^*, \Omega_n, \Omega_{n+1}) &= \frac{1}{\kappa^*} \prod_{A=1}^4 \frac{\partial}{\partial \tilde{\eta}_p^A} \left[ \frac{\delta^4(k + \lambda_n \tilde{\lambda}_n + \lambda_{n+1} \tilde{\lambda}_{n+1}) \delta^8(\lambda_p \tilde{\eta}_p + \lambda_n \tilde{\eta}_n + \lambda_{n+1} \tilde{\eta}_{n+1})}{\langle p n \rangle \langle n n+1 \rangle \langle n+1 p \rangle} \right] \\ &= \frac{\delta^4(k + \lambda_n \tilde{\lambda}_n + \lambda_{n+1} \tilde{\lambda}_{n+1})}{\kappa^*} \frac{\delta^4(\tilde{\eta}_n \langle p n+1 \rangle + \tilde{\eta}_{n+1} \langle p n \rangle)}{\langle p n \rangle \langle n n+1 \rangle \langle n+1 p \rangle}. \end{aligned} \quad (4.44)$$

Here  $p$  is the off-shell gluon direction and  $\kappa^*$  was defined in Section 2 when introducing  $k_T$  decomposition of the off-shell gluon momentum  $k$ . It should be noted, that each of  $\mathcal{V}$  operators above could be exchanged for  $\tilde{\mathcal{V}}$  operator, so that this vertex operator representation is not unique. Note also, that the ambitwistor string vertex operator we got is non-local, what is expected as Wilson line is non-local object itself. Performing integrations<sup>13</sup> over helicity spinors  $\lambda_i, \tilde{\lambda}_i$  we get (here and below we always assume the action of the projection operator  $\partial_{\tilde{\eta}_p}^4$  acting on  $\mathcal{V}_{n,n+1}^{\text{WL}}$  and all correlation functions containing it)

$$\mathcal{V}_{n,n+1}^{\text{WL}} = \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d\beta_2}{\beta_2} \int \frac{d\beta_1}{\beta_1} \frac{1}{\beta_1^2 \beta_2} \mathcal{V}_n \mathcal{V}_{n+1} \Big|_{T^a T^b \rightarrow i f^{abc} T^c \rightarrow T^c}, \quad (4.45)$$

where

$$\lambda_n = \underline{\lambda}_n + \beta_2 \underline{\lambda}_{n+1}, \quad \tilde{\lambda}_n = \beta_1 \tilde{\lambda}_n + \frac{(1 + \beta_1)}{\beta_2} \tilde{\lambda}_{n+1}, \quad \tilde{\eta}_n = -\beta_1 \tilde{\eta}_n, \quad (4.46)$$

$$\lambda_{n+1} = \underline{\lambda}_{n+1} + \frac{(1 + \beta_1)}{\beta_1 \beta_2} \underline{\lambda}_n, \quad \tilde{\lambda}_{n+1} = -\beta_1 \tilde{\lambda}_{n+1} - \beta_1 \beta_2 \tilde{\lambda}_n, \quad \tilde{\eta}_{n+1} = \beta_1 \beta_2 \tilde{\eta}_n. \quad (4.47)$$

with

$$\underline{\lambda}_n = \lambda_p, \quad \tilde{\lambda}_n = \frac{\langle \xi | k}{\langle \xi p \rangle}, \quad \tilde{\eta}_n = \tilde{\eta}_p; \quad \underline{\lambda}_{n+1} = \lambda_\xi, \quad \tilde{\lambda}_{n+1} = \frac{\langle p | k}{\langle \xi p \rangle}, \quad \tilde{\eta}_{n+1} = 0, \quad (4.48)$$

where  $\lambda_\xi \equiv \langle \xi |$  is some arbitrary spinor. It is useful to identify it with the spinor  $\lambda_q$  coming from helicity spinor decomposition of auxiliary vector  $q$  arising in  $k_T$  decomposition of off-shell gluon momentum  $k$ .

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<sup>13</sup>The details of this procedure could be found in Appendix A.

The off-shell amplitude with one off-shell and  $n$  on-shell legs is then given by the following ambitwistor string correlation function:

$$A_{k,n+1}^* = \left\langle \tilde{\mathcal{V}}_1 \dots \tilde{\mathcal{V}}_k \mathcal{V}_{k+1} \dots \mathcal{V}_n \mathcal{V}_{n+1,n+2}^{\text{WL}} \right\rangle. \quad (4.49)$$

Evaluating first ambitwistor string correlator of on-shell vertexes with the help of (4.39) we get

$$\begin{aligned} A_{k,n+1}^* &= \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d\beta_2}{\beta_2} \int \frac{d\beta_1}{\beta_1} \frac{1}{\beta_1^2 \beta_2} \frac{1}{\text{Vol GL}(2, \mathbb{C})} \\ &\times \int \prod_{a=1}^{n+2} \frac{ds_a d\sigma_a}{s_a(\sigma_a - \sigma_{a+1})} \prod_{p=k+1}^{n+2} \bar{\delta}^2(\lambda_p - s_p \lambda(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|4}(\tilde{\lambda}_i - s_i \tilde{\lambda}(\sigma_i), \tilde{\eta}_i - s_i \tilde{\chi}(\sigma_i)). \end{aligned} \quad (4.50)$$

Now introducing unity decomposition in the form [26]:

$$1 = \frac{1}{\text{Vol GL}(k)} \int d^{k \times (n+2)} C d^{k \times k} L (\det L)^{n+2} \delta^{k \times (n+2)} (C - L \cdot C^V[s, \sigma]), \quad (4.51)$$

where the integral over  $L$  matrix is an integral over  $\text{GL}(k)$  linear transformations and  $C^V[\sigma]$  denotes the Veronese map from  $(\mathbb{C}^2)^{n+2}/\text{GL}(2)$  to  $Gr(k, n+2)$  Grassmannian [26] (see also [44]):

$$C^V[s, \sigma] = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \sigma^V[s_1, \sigma_1] & \sigma^V[s_2, \sigma_2] & \cdots & \sigma^V[s_{n+2}, \sigma_{n+2}] \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}, \quad \sigma^V[s, \sigma] \equiv \begin{pmatrix} \xi \\ \xi \sigma \\ \vdots \\ \xi \sigma^{k-1} \end{pmatrix}, \quad (4.52)$$

where [4, 44] :

$$\xi_i = s_i^{-1} \prod_{j=1, j \neq i}^k (\sigma_j - \sigma_i)^{-1}, \quad i \in (1, k) \quad (4.53)$$

$$\xi_i = s_i \prod_{j=1}^k (\sigma_j - \sigma_i)^{-1}, \quad i \in (k+1, n+2) \quad (4.54)$$

Integrating (4.50) over the  $s_a$  and  $\sigma_a$  we get

$$\begin{aligned} A_{k,n+1}^* &= \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d\beta_2}{\beta_2} \int \frac{d\beta_1}{\beta_1} \frac{1}{\beta_1^2 \beta_2} \frac{1}{\text{Vol GL}(k)} \\ &\times \int d^{k \times (n+2)} C F(C) \delta^{k \times 2}(C \cdot \tilde{\lambda}) \delta^{k \times 4}(C \cdot \tilde{\eta}) \delta^{(n+2-k) \times 2}(C^\perp \cdot \lambda), \end{aligned} \quad (4.55)$$

where

$$F(C) = \int \frac{1}{\text{Vol } GL(2, \mathbb{C})} \prod_{a=1}^{n+2} \frac{ds_a d\sigma_a}{s_a(\sigma_a - \sigma_{a+1})} d^{k \times k} L \delta^{k \times (n+2)} (C - L \cdot C^V[s, \sigma]), \quad (4.56)$$

and

$$\begin{aligned} \delta^{k \times 2}(C \cdot \tilde{\lambda}) &\equiv \prod_{a=1}^k \delta^2 \left( \sum_{i=1}^n c_{ai} \tilde{\lambda}_i \right), \quad \delta^{(n+2-k) \times 2}(C^\perp \cdot \lambda) \equiv \prod_{b=k+1}^{n+2} \delta^2 \left( \sum_{j=1}^{n+2} c_{bj}^\perp \lambda_j \right), \\ \delta^{k \times 4}(C \cdot \tilde{\eta}) &\equiv \prod_{a=1}^k \delta^4 \left( \sum_{i=1}^{n+2} c_{ai} \tilde{\eta}_i \right), \end{aligned} \quad (4.57)$$

$C^\perp$  matrix is defined by identity  $C \cdot (C^\perp)^T = 0$  and it is assumed that all matrix manipulations are performed after  $GL(k)$  gauge fixing. The delta functions above should be thought as  $\delta(x) = 1/x$ , so that the corresponding contour integral computes the residue at  $x = 0$  [96].

Next, by construction  $F(C)$  contains  $(k-2) \times (n-k)$  delta function factors forcing integral over  $C$ 's to have Veronese form [26]. In general  $F(C)$  is rather complicated rational function of minors of  $C$  matrix, see the discussion in Section 5. However, it could be shown that the choice of  $F(C)$  in the form

$$F(C) = \frac{1}{(1 \cdots k)(2 \cdots k+1) \cdots (n+2 \cdots k-1)}. \quad (4.58)$$

correctly reproduces the results of subsequent integration over  $C$  matrix. Here we use standard notations  $(i_1 \dots i_k)$  to denote minors of  $C$  matrix constructed from columns of  $C$  with numbers  $i_1, \dots, i_k$ <sup>14</sup>. Taking next integrations over  $\beta_1, \beta_2$  variables in (4.55) as explained in detail in Appendix A we get

$$A_{k,n+1}^* = \int_{\Gamma_{k,n+2}^{tree}} \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \text{Reg.} \frac{\delta^{k \times 2}(C \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4}(C \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2}(C^\perp \cdot \underline{\lambda})}{(1 \cdots k) \cdots (n+1 \cdots k-2)(n+2 \ 1 \cdots k-1)}, \quad (4.59)$$

with

$$\text{Reg.} = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(n+2 \ 1 \cdots k-1)}{(n+1 \ 1 \cdots k-1)}. \quad (4.60)$$

in a complete agreement with our previous derivation [51]. Here we have also introduced

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<sup>14</sup>We hope there will be no confusion with previous definition  $(ij) = \sigma_{ia} \sigma_j^\alpha$  used in  $d^2 \sigma_a$  integrals over homogeneous coordinates on Riemann sphere.

the following notation

$$\begin{aligned}
\underline{\underline{\lambda}}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\underline{\lambda}}_{n+1} &= \lambda_p, & \underline{\underline{\lambda}}_{n+2} &= \xi \\
\underline{\underline{\lambda}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\underline{\lambda}}_{n+1} &= \frac{\langle \xi | k}{\langle \xi p \rangle}, & \underline{\underline{\lambda}}_{n+2} &= -\frac{\langle p | k}{\langle \xi p \rangle}, \\
\underline{\underline{\eta}}_i &= \tilde{\eta}_i, & i &= 1, \dots, n, & \underline{\underline{\eta}}_{n+1} &= \tilde{\eta}_p, & \underline{\underline{\eta}}_{n+2} &= 0.
\end{aligned} \tag{4.61}$$

We will make some comments on the choice of integration contour  $\Gamma_{k,n+2}^{tree}$  in Section 5.

Finally, performing the inverse operation - that is taking partial integrations and reducing integral in (5.99) to the integral over  $Gr(2, n+2)$  Grassmannian<sup>15</sup>, we get (the details could be found in next section) :

$$A_{k,n+1}^* = \int \prod_{a=1}^{n+2} \frac{d^2 \sigma_a}{(a \ a + 1)} \frac{Reg.^V}{\text{Vol GL}(2, \mathbb{C})} \prod_{p=k+1}^{n+2} \bar{\delta}^2(\underline{\underline{\lambda}}_p - \underline{\underline{\lambda}}(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|4}(\underline{\underline{\lambda}}_i - \underline{\underline{\lambda}}(\sigma_i), \underline{\underline{\eta}}_i - \underline{\underline{\chi}}(\sigma_i)), \tag{4.62}$$

where  $Reg.^V$  factor is given by

$$Reg.^V = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(k \ n + 1)}{(k \ n + 2)} \tag{4.63}$$

and doubly underlined functions are defined as

$$(\underline{\underline{\lambda}}, \underline{\underline{\mu}}, \underline{\underline{\chi}}) = \sum_{i=1}^k \frac{(\underline{\lambda}_i, 0, 0)}{(\sigma \ \sigma_i)}, \quad (\underline{\underline{\mu}}, \underline{\underline{\lambda}}, \underline{\underline{\chi}}) = \sum_{p=k+1}^{n+2} \frac{(0, \underline{\lambda}_p, \underline{\eta}_p)}{(\sigma \ \sigma_p)}. \tag{4.64}$$

The result for the case of amplitudes with multiple off-shell legs  $A_{k,m+n}^*$  could be obtained along the same lines, see section 6 for example of gluing procedure for off-shell amplitude with multiple off-shell legs. In the case with first  $m$  particles on-shell and last  $n$  being off-shell we would get for  $A_{k,m+n}^*$ :

$$\int \prod_{a=1}^{n+2} \frac{d^2 \sigma_a}{(a \ a + 1)} \frac{Reg.^V(m+1, \dots, m+n)}{\text{Vol GL}(2, \mathbb{C})} \prod_{p=k+1}^{m+2n} \bar{\delta}^2(\underline{\underline{\lambda}}_p - \underline{\underline{\lambda}}(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|4}(\underline{\underline{\lambda}}_i - \underline{\underline{\lambda}}(\sigma_i), \underline{\underline{\eta}}_i - \underline{\underline{\chi}}(\sigma_i)), \tag{4.65}$$

where

$$Reg.^V(m+1, \dots, m+n) = \prod_{j=1}^n Reg.^V(j+m), \quad Reg.^V(j+m) = \frac{\langle \xi_j p_j \rangle}{\kappa_j^*} \frac{(k \ 2j - 1 + m)}{(k \ 2j + m)}. \tag{4.66}$$

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<sup>15</sup>The  $Gr(2, n+2)$  Grassmannian is embedded into  $Gr(k, n+2)$  Grassmannian again with the help of Veronese map, see for example [44].

and external kinematical variables are defined as

$$\begin{aligned}
\underline{\lambda}_i &= \lambda_i, & i &= 1, \dots, m, & \underline{\lambda}_{m+2j-1} &= \lambda_{p_j}, & \underline{\lambda}_{m+2j} &= \xi_j, & j &= 1, \dots, n, \\
\underline{\tilde{\lambda}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, m, & \underline{\tilde{\lambda}}_{m+2j-1} &= \frac{\langle \xi_j | k_{m+j} \rangle}{\langle \xi_j p_j \rangle}, & \underline{\tilde{\lambda}}_{m+2j} &= -\frac{\langle p_j | k_{m+j} \rangle}{\langle \xi_j p_j \rangle}, & j &= 1, \dots, n, \\
\underline{\tilde{\eta}}_i &= \tilde{\eta}_i, & i &= 1, \dots, m, & \underline{\tilde{\eta}}_{m+2j-1} &= \tilde{\eta}_{p_j}, & \underline{\tilde{\eta}}_{m+2j} &= 0, & j &= 1, \dots, n.
\end{aligned} \tag{4.67}$$

As in the case with one off-shell leg it is possible to rewrite (4.65) as an integral over  $Gr(k, m+2n)$  Grassmannian making identical assumptions about the form of  $F(C)$  function. The result is then given by the following Grassmannian integral first conjectured in [52]:

$$\begin{aligned}
A_{k, m+n}^* &= \int_{\Gamma_{k, m+2n}^{tree}} \frac{d^{k \times (m+2n)} C}{\text{Vol}[GL(k)]} \text{Reg.}(m+1, \dots, m+n) \times \\
&\times \frac{\delta^{k \times 2} (C \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C \cdot \underline{\tilde{\eta}}) \delta^{(m+2n-k) \times 2} (C^\perp \cdot \underline{\lambda})}{(1 \dots k) \dots (m \dots m+k-1) (m+1 \dots m+k) \dots (m+2n \dots k-1)},
\end{aligned} \tag{4.68}$$

where the external kinematical variables are chosen as in (4.67) and  $\text{Reg.}(m+1, \dots, m+n)$  function is given by the products of ratios of minors of  $C$  matrix:

$$\begin{aligned}
\text{Reg.}(m+1, \dots, m+n) &= \prod_{j=1}^n \text{Reg.}(j+m), \\
\text{Reg.}(j+m) &= \frac{\langle \xi_j p_j \rangle}{\kappa_j^*} \frac{(2j+m \quad 2j+1+m \dots 2j+k-1+m)}{(2j-1+m \quad 2j+1+m \dots 2j+k-1+m)}.
\end{aligned} \tag{4.69}$$

At the end, we want to stress that the explicit form of (4.63) and (4.66) is not unique and is in fact related to the  $GL(k)$  gauge choice in (4.68) and (5.99).

## 4.2 Stress-tensor operator supermultiplet

The ambitwistor string vertex operators for operators from stress-tensor operator supermultiplet could be constructed similar to the case of Wilson line vertex operator. This way the corresponding string vertex operator is given by<sup>16</sup>

$$\mathcal{V}_{n, n+1}^{\text{ST}} = \int \prod_{i=n}^{n+1} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i}{\text{Vol}[GL(1)]} d^4 \tilde{\eta}_i F_{2,2}(\Omega_n, \Omega_{n+1}; \mathcal{T}) \Big|_{\lambda \rightarrow -\lambda} \mathcal{V}_n \mathcal{V}_{n+1} \Big|_{T^a T^b \rightarrow \delta^{ab} \rightarrow 1}, \tag{4.70}$$

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<sup>16</sup>In this case also each of  $\mathcal{V}$  vertex operators could be exchanged with  $\tilde{\mathcal{V}}$  vertex operator.



where the minimal form factor  $F_{2,2}(\Omega_n, \Omega_{n+1}; \mathcal{T})$  is given by [48]:

$$F_{2,2}(\Omega_n, \Omega_{n+1}; \mathcal{T}) = \delta^2(\tilde{\underline{\lambda}}_{n+1}) \delta^4(\tilde{\underline{q}}_{n+1}) \delta^2(\tilde{\underline{\lambda}}_{n+2}) \delta^4(\tilde{\underline{q}}_{n+2}) \quad (4.71)$$

with ( $q$  and  $\gamma^-$  are the operator's momentum and supermomentum correspondingly)

$$\begin{aligned} \tilde{\underline{\lambda}}_{n+1} &= \tilde{\lambda}_{n+1} - \frac{\langle n+2|q}{\langle n+2 \ n+1 \rangle}, & \tilde{\underline{q}}_{n+1}^- &= \tilde{q}_{n+1}^- - \frac{\langle n+2|\gamma^-}{\langle n+2 \ n+1 \rangle}, & \tilde{\underline{q}}_{n+1}^+ &= \tilde{q}_{n+1}^+, \\ \tilde{\underline{\lambda}}_{n+2} &= \tilde{\lambda}_{n+2} - \frac{\langle n+1|q}{\langle n+1 \ n+2 \rangle}, & \tilde{\underline{q}}_{n+2}^- &= \tilde{q}_{n+2}^- - \frac{\langle n+1|\gamma^-}{\langle n+1 \ n+2 \rangle}, & \tilde{\underline{q}}_{n+2}^+ &= \tilde{q}_{n+2}^+. \end{aligned} \quad (4.72)$$

Integrating over helicity spinors  $\lambda_i, \tilde{\lambda}_i$  we get

$$\mathcal{V}_{n,n+1}^{\text{ST}} = -\langle \xi_A \xi_B \rangle^2 \int d\beta_1 \int d\beta_2 \mathcal{V}_n \mathcal{V}_{n+1} \Big|_{T^a T^b \rightarrow \delta^{ab} \rightarrow 1}, \quad (4.73)$$

where

$$\lambda_{n+1} = \xi_A - \beta_1 \xi_B, \quad \tilde{\lambda}_{n+1} = \frac{1}{\beta_1 \beta_2 - 1} \frac{\langle \xi_B | q}{\langle \xi_B \xi_A \rangle} + \frac{\beta_2}{\beta_1 \beta_2 - 1} \frac{\langle \xi_A | q}{\langle \xi_A \xi_B \rangle} \quad (4.74)$$

$$\lambda_{n+2} = \xi_B - \beta_2 \xi_A, \quad \tilde{\lambda}_{n+2} = \frac{1}{\beta_1 \beta_2 - 1} \frac{\langle \xi_A | q}{\langle \xi_A \xi_B \rangle} + \frac{\beta_1}{\beta_1 \beta_2 - 1} \frac{\langle \xi_B | q}{\langle \xi_B \xi_A \rangle} \quad (4.75)$$

and

$$\tilde{\eta}_{n+1}^- = \frac{1}{\beta_1 \beta_2 - 1} \frac{\langle \xi_B | \gamma^-}{\langle \xi_B \xi_A \rangle} + \frac{\beta_2}{\beta_1 \beta_2 - 1} \frac{\langle \xi_A | \gamma^-}{\langle \xi_A \xi_B \rangle}, \quad \tilde{\eta}_{n+1}^+ = 0 \quad (4.76)$$

$$\tilde{\eta}_{n+2}^- = \frac{1}{\beta_1 \beta_2 - 1} \frac{\langle \xi_A | \gamma^-}{\langle \xi_A \xi_B \rangle} + \frac{\beta_1}{\beta_1 \beta_2 - 1} \frac{\langle \xi_B | \gamma^-}{\langle \xi_B \xi_A \rangle}, \quad \tilde{\eta}_{n+2}^+ = 0 \quad (4.77)$$

Evaluation of string correlation function with stress-tensor vertex operator insertion closely follows the corresponding calculation for the case of Wilson line vertex operator insertion presented above. As a result we obtain the following form factor representation<sup>17</sup>

$$\begin{aligned} F_{k,n} &= \int \prod_{a=1}^{n+2} \frac{d^2 \sigma_a}{(a \ a+1) \text{Vol GL}(2, \mathbb{C})} \frac{\text{Reg.}}{\prod_{p=k+1}^{n+2} \delta^2(\underline{\lambda}_p - \underline{\lambda}(\sigma_p))} \prod_{i=1}^k \delta^{2|4}(\underline{\tilde{\lambda}}_i - \underline{\tilde{\lambda}}(\sigma_i), \underline{\underline{q}}_i - \underline{\underline{q}}(\sigma_i)) \\ &+ \text{ other gluing positions,} \end{aligned} \quad (4.78)$$

where *Reg.* factor is now given by [44];

$$\text{Reg.} = \langle \xi_A \xi_B \rangle^2 \frac{Y}{1-Y}, \quad Y = \prod_{j=n+2-k}^n \frac{(j \ n+1)}{(j \ n+2)} \prod_{i=1}^{k-1} \frac{(n+2 \ i)}{(n+1 \ i)} \quad (4.79)$$

---

<sup>17</sup>It was already obtained in [44] applying Veronese map to the Grassmannian integral representation of [48].

and doubly underlined functions are defined as in the case of Wilson line insertion:

$$\left(\underline{\underline{\lambda}}, \underline{\underline{\mu}}, \underline{\underline{\chi}}\right) = \sum_{i=1}^k \frac{(\underline{\underline{\lambda}}_i, 0, 0)}{(\sigma \sigma_i)}, \quad \left(\underline{\underline{\tilde{\mu}}}, \underline{\underline{\tilde{\lambda}}}, \underline{\underline{\tilde{\chi}}}\right) = \sum_{p=k+1}^{n+2} \frac{(0, \underline{\underline{\tilde{\lambda}}}_p, \underline{\underline{\tilde{\mu}}}_p)}{(\sigma \sigma_p)}, \quad (4.80)$$

where

$$\begin{aligned} \underline{\underline{\lambda}}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\underline{\lambda}}_{n+1} &= \xi_A, & \underline{\underline{\lambda}}_{n+2} &= \xi_B \\ \underline{\underline{\tilde{\lambda}}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\underline{\tilde{\lambda}}}_{n+1} &= -\frac{\langle \xi_B | q}{\langle \xi_B \xi_A \rangle}, & \underline{\underline{\tilde{\lambda}}}_{n+2} &= -\frac{\langle \xi_A | q}{\langle \xi_A \xi_B \rangle}, \\ \underline{\underline{\tilde{\mu}}}_i^+ &= \tilde{\mu}_i^+, & i &= 1, \dots, n, & \underline{\underline{\tilde{\mu}}}_{n+1}^+ &= 0, & \underline{\underline{\tilde{\mu}}}_{n+2}^+ &= 0, \\ \underline{\underline{\tilde{\mu}}}_i^- &= \tilde{\mu}_i^-, & i &= 1, \dots, n, & \underline{\underline{\tilde{\mu}}}_{n+1}^- &= -\frac{\langle \xi_B | \gamma^-}{\langle \xi_B \xi_A \rangle}, & \underline{\underline{\tilde{\mu}}}_{n+2}^- &= -\frac{\langle \xi_A | \gamma^-}{\langle \xi_A \xi_B \rangle}. \end{aligned} \quad (4.81)$$

In the formula above the term + *other gluing positions* denotes all other gluing positions of minimal form factor to the color ordered on-shell amplitude. Note, that the original string correlation function contains all these terms corresponding to different gluing positions from the very beginning.

## 5 Grassmannians, scattering equations and link representations

In this section we want to discuss the connection between different representations of tree level form factors, such as Grassmannian integral, scattering equations and link representations as well as discuss issues related to the explicit choice of  $F(C)$  function.

Let us start with the previously obtained Grassmannian integral representations for form factors of operators from stress-tensor operator supermultiplet [47–50] and Wilson line operator insertions [51, 52]. The corresponding Grassmannian integral representations could be written as<sup>18</sup>

$$\int \frac{d^{k \times (n+2)} C \, d^{2k} \rho}{\text{Vol}[GL(k)]} \text{Reg.} \frac{\delta^{k \times 2} (C \cdot \underline{\underline{\lambda}}) \delta^{k \times 4} (C \cdot \underline{\underline{\tilde{\eta}}}) \delta^{(n+2) \times 2} (\rho \odot C - \underline{\underline{\lambda}})}{(1 \dots k)(2 \dots k+1) \dots (n+2 \dots k-1)}, \quad (5.82)$$

where in the case of stress-tensor operator supermultiplet [48]:

$$\text{Reg.} = \langle \xi_A \xi_B \rangle^2 \frac{Y}{1-Y}, \quad Y = \frac{(n-k+2 \dots n \, n+1)(n+2 \, 1 \dots k-1)}{(n-k+2 \dots n \, n+2)(n+1 \, 1 \dots k-1)}. \quad (5.83)$$

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<sup>18</sup>In the case of form factors of operators from stress-tensor operator supermultiplet we should also sum over different top-cell forms corresponding to different gluing positions of minimal form factor.

and kinematical data are given by (4.81). In the case of single Wilson line insertion we have [51] (the Grassmannian integral representation for the case of multiple Wilson lines insertions is given by (4.68), see also [52]):

$$Reg. = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(n+2 \ 1 \cdots k-1)}{(n+1 \ 1 \cdots k-1)}. \quad (5.84)$$

with kinematical data defined in (4.61). Here,  $\rho_I^\alpha$  is a  $k \times 2$  matrix, while  $\odot$  and  $\cdot$  denote contractions with  $I$  ( $I \in 1, \dots, k$ ) and  $a$  ( $a \in 1, \dots, n+2$ ) indices of  $C_{Ia}$  matrix correspondingly. The scattering equations representations for the above form factors could be easily obtained via Veronese map [26, 44]. As we already mentioned before, the Veronese map is an embedding of  $Gr(2, n+2)$  into  $Gr(k, n+2)$  Grassmannian (4.52), assuming that  $F(C)$  can be chosen in the form (4.58). Using resolution of unity (4.51), fixing  $GL(k)$  gauge to enforce rational form of scattering equations [11] and performing integration over  $C$  matrix the original Grassmannian representation takes the form:

$$\int \prod_{a=1}^{n+2} \frac{d^2 \sigma_a}{(a \ a+1)} \frac{Reg.^V}{\text{Vol } GL(2, \mathbb{C})} \prod_{p=k+1}^{n+2} \bar{\delta}^2(\underline{\lambda}_p - \underline{\lambda}(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|4}(\underline{\tilde{\lambda}}_i - \underline{\tilde{\lambda}}(\sigma_i), \underline{\eta}_i - \underline{\tilde{\chi}}(\sigma_i)), \quad (5.85)$$

where we have also performed transition to homogeneous coordinates on Riemann sphere and introduced doubly underlined functions as in previous section. Without fixing  $GL(k)$  gauge and not taking integrations over  $\rho$  matrix we would obtain polynomial form of the resulting scattering equations as in [97]. The Veronese map of  $Reg.$  functions is given by<sup>19</sup>:

$$Reg.^V = \langle \xi_A \xi_B \rangle^2 \frac{Y}{1-Y}, \quad Y = \prod_{j=n+2-k}^n \frac{(j \ n+1)}{(j \ n+2)} \prod_{i=1}^{k-1} \frac{(n+2 \ i)}{(n+1 \ i)} \quad (5.86)$$

for the case of stress-tensor supermultiplet and by

$$Reg.^V = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(k \ n+1)}{(k \ n+2)} \quad (5.87)$$

in the case of single Wilson line insertion. The results for the case of multiple Wilson line insertions could be obtained along the same lines. For example, in the case with first  $m$  particles on-shell and last  $n$  being Wilson line insertions we would get

$$\int \prod_{a=1}^{n+2} \frac{d^2 \sigma_a}{(a \ a+1)} \frac{Reg.^V(m+1, \dots, m+n)}{\text{Vol } GL(2, \mathbb{C})} \prod_{p=k+1}^{m+2n} \bar{\delta}^2(\underline{\lambda}_p - \underline{\lambda}(\sigma_p)) \prod_{i=1}^k \bar{\delta}^{2|4}(\underline{\tilde{\lambda}}_i - \underline{\tilde{\lambda}}(\sigma_i), \underline{\eta}_i - \underline{\tilde{\chi}}(\sigma_i)), \quad (5.88)$$

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<sup>19</sup>It is easily obtained with the use of Vandermonde determinant formula.

where

$$Reg.^V(m+1, \dots, m+n) = \prod_{j=1}^n Reg.^V(j+m), \quad Reg.^V(j+m) = \frac{\langle \xi_j p_j \rangle}{\kappa_j^*} \frac{(k 2j - 1 + m)}{(k 2j + m)}. \quad (5.89)$$

and external kinematical variables are defined as in (4.67)

The link representation in its turn could be easily obtained from the scattering equations formulae. To see that, let's start for example with (4.39). Introducing resolutions of unities in the form

$$1 = \int dc_{ai} \delta \left( c_{ai} - \frac{s_a s_i}{\sigma_a - \sigma_i} \right) \quad (5.90)$$

we immediately get

$$\begin{aligned} A_{k,n+2} &= \int \prod_{\substack{b \in f \\ j \in g}} dc_{bj} \prod_{a \in f} \delta^{2|4} \left( \tilde{\lambda}_a - \sum_{i \in g} c_{ai} \tilde{\lambda}_i \middle| \tilde{\eta}_a - \sum_{i \in g} c_{ai} \tilde{\eta}_i \right) \times \\ &\times \prod_{i \in g} \delta^2 \left( \lambda_i - \sum_{a \in f} c_{ai} \lambda_a \right) F(C), \end{aligned} \quad (5.91)$$

with

$$F(C) = \int \frac{1}{\text{Vol } GL(2, \mathbb{C})} \prod_{b=1}^n \frac{ds_b d\sigma_b}{s_a(\sigma_b - \sigma_{b+1})} \prod_{\substack{a \in f \\ i \in g}} \delta \left( c_{ai} - \frac{s_a s_i}{\sigma_a - \sigma_i} \right). \quad (5.92)$$

and  $f, g$  denoting index sets  $f = 1, \dots, k$  and  $g = k+1, \dots, n+2$ . Auxiliary variables  $c_{ai}$  and equation (5.91) are usually called *link variables* [98] and link representation of  $A_{k,n+2}$  amplitude correspondingly. Note that in this representation the kinematical constraints become linear in terms of  $c_{ai}$ , while all nonlinearity is accumulated in kinematically independent function  $F(C)$ .

Now, let us discuss the questions related to the explicit form of  $F(C)$  function and present arguments in favor of the statement that  $F(C)$  can be always chosen in the form (4.58). It is easy to see that the integration over all  $c_{ai}$  variables is in fact identical to the integration over  $Gr(k, n)$  Grassmannian for the specific choice of  $GL(k)$  gauge with first  $k$  columns of  $C$  matrix fixed to be equal to unit matrix. The latter gauge choice also determines the index sets  $f$  and  $g$ . Namely,  $f$  contains the numbers of columns constituting unity matrix. The different choices of  $f$  and  $g$  sets with given total numbers of elements  $\#f = k$ ,  $\#g = n - k + 2$  in each set correspond to different gauge choices and to different rearrangements of  $\mathcal{V}_a$  and  $\tilde{\mathcal{V}}_a$  vertex operator among themselves in correlation

function. Of course all gauge should lead to the same result. In previous considerations we had  $f = (1, \dots, k)$  and  $g = (k + 1, \dots, n + 2)$ . One can also consider the expression without gauge fixing and we have already used such expressions in the previous section. To get it, we may formally replace

$$\int \prod_{\substack{b \in f \\ j \in g}} dc_{bj} = \int \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]}. \quad (5.93)$$

Next, it is convenient to rearrange delta functions of kinematical constraints in the form [22] :

$$\begin{aligned} & \int \prod_{\substack{b \in f \\ j \in g}} dc_{bj} \prod_{a \in f} \delta^{2|0} \left( \tilde{\lambda}_a - \sum_{i \in g} c_{ai} \tilde{\lambda}_i \right) \prod_{i \in g} \delta^2 \left( \lambda_i - \sum_{a \in f} c_{ai} \lambda_a \right) = \\ & = \delta^4 \left( \sum_{j=1}^{n+2} \lambda_j \tilde{\lambda}_j \right) J(\lambda, \tilde{\lambda}) \int d^{(k-2)(n-k)} \tau_A \prod_{\substack{a \in f \\ i \in g}} \delta(c_{ai} - c_{ai}(\tau|kin.)), \end{aligned} \quad (5.94)$$

where  $J(\lambda, \tilde{\lambda})$  is the Jacobian of transformation and  $c_{ai}(\tau|kin.)$  is a general solution of underdetermined system of linear equations [22, 99]

$$\begin{aligned} c_{ai} \lambda_a &= -\lambda_i, \\ c_{ai} \tilde{\lambda}_i &= -\tilde{\lambda}_a, \end{aligned} \quad (5.95)$$

with  $a \in f$ ,  $i \in g$ . The solution depends on extremal kinematical data  $\lambda_i, \tilde{\lambda}_i$  as well as on the arbitrary  $(k-2)(n-k)$  parameters  $\tau_A$ . The explicit form of  $c_{ai}(\tau|kin.)$  for general  $n$  and  $k$  can be found in [99]. For example, for  $n+2=6$ ,  $k=3$  and  $f \in (1, 3, 5)$  and  $g \in (2, 4, 6)$  we have  $c_{ai}(\tau|kin.) = c_{ai}^* + \epsilon_{aa_1 a_2} \epsilon_{ii_1 i_2} \langle a_1 a_2 \rangle [i_1 i_2] \tau$  [22], where  $c_{ai}^*$  is some particular solution of (5.95).

Using the representation (5.94) we can remove integration over  $d^{k \times (n+2)} C / \text{Vol}[GL(k)]$ . Next, lets for simplicity fix helicities of external particles in such a way that Grassmann delta functions  $\delta^{0|4}$  go to  $1^{20}$ . All these manipulations reduce our initial expression for the correlation function of vertex operators (4.39) to

$$A_{k,n+2} = \delta^4 \left( \sum_{j=1}^{n+2} \lambda_j \tilde{\lambda}_j \right) J(\lambda, \tilde{\lambda}) \int_{\Gamma} d^{(k-2)(n-k)} \tau_A F(C) \Big|_{c_{ai} \mapsto c_{ai}(\tau|kin.)}, \quad (5.96)$$

with the appropriate choice of integration contour  $\Gamma$ .

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<sup>20</sup>This is always possible for appropriate  $GL(k)$  gauge and external state choices. For example, for  $n+2=6$ ,  $k=3$  and  $f=2, 4, 6$ ,  $g=1, 3, 5$  the appropriate choice of the external particles helicities will be  $(+ - + - + -)$  [22].

Next, the function  $F(C)$  could be evaluated for general values of  $n$  and  $k$  in terms of matrix elements  $c_{ai}$  [99]. It is a rather complicated expression. The most studied case is  $k = 3$  [4, 26, 100] and it is believed that for  $k > 3$  the behavior will be essentially the same as in  $k = 3$  case [26]. Let us also concentrate on the  $k = 3$  case as representative, yet simple enough example. In this case we can rewrite  $F(C)$  function in terms of minors of  $C$  matrix and get [26, 100]

$$F^{k=3}(C) = H(C) \frac{1}{S_6 \dots S_{n+2}}, \quad H(C) = \frac{\prod_{j=6}^{n+1} (12j)(23j-1) \prod_{i=5}^{n+1} (13i)}{(n+1n+21)(123)(234)}, \quad (5.97)$$

and  $(j = 6, \dots, n+2)$

$$S_j = (j-2j-1j)(j12)(23j-2)(j-113) - (j-1j1)(123)(3j-2j-1)(j2j-2). \quad (5.98)$$

On the other hand, if we consider representation of the amplitude  $A_{k,n+2}$  in terms of the integral over Grassmannian  $Gr(k, n+2)$

$$A_{k,n+2} = \int_{\Gamma_{k,n+2}^{tree}} \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \frac{\delta^{k \times 2} (C \cdot \tilde{\underline{\lambda}}) \delta^{k \times 4} (C \cdot \tilde{\underline{\eta}}) \delta^{(n+2-k) \times 2} (C^\perp \cdot \underline{\lambda})}{(1 \dots k) \dots (n+1 \dots k-2)(n+2 \ 1 \dots k-1)}, \quad (5.99)$$

and use manipulations as before we will arrive at a similar expression (5.96) [26], but with different expression for  $F(C)$  function, which we will denote now as  $F_{Gr}(C)$ :

$$A_{n+2,k} = \delta^4 \left( \sum_{j=1}^{n+2} \lambda_j \tilde{\lambda}_j \right) J(\lambda, \tilde{\lambda}) \int_{\Gamma_{k,n+2}^{tree}} d^{(k-2)(n-k)} \tau_A F_{Gr}(C) \big|_{c_{ai} \mapsto c_{ai}(\tau|kin.)} \quad (5.100)$$

where for  $k = 3$

$$F_{Gr}^{k=3}(C) = \tilde{H}(C) \frac{1}{\tilde{S}_6 \dots \tilde{S}_{n+2}}, \quad \tilde{H}(C) = \frac{\prod_{j=6}^{n+1} (12j)(23j-1)}{(n+1n+21)(123)(234)}, \quad (5.101)$$

and

$$\tilde{S}_j = (j-2j-1j)(j12)(23j-2), \quad j = 6, \dots, n+2. \quad (5.102)$$

The  $F_{Gr}^{k=3}(C)$  function is given by essentially rearranged cyclic factor [26]:

$$F_{Gr}^{k=3}(C) = \frac{1}{(123)(234) \dots (n+212)} \quad (5.103)$$

One can consider  $S_j$  or  $\tilde{S}_j$  function as the explicit construction of the map  $\mathbf{S} = (\tilde{S}_6, \dots, \tilde{S}_{n+2})$ ,  $\mathbf{S} : \mathbb{C}^{(n-3)} \mapsto \mathbb{C}^{(n-3)}$ , which zeros determine the integration contour  $\Gamma_{3,n+2}^{tree}$ . It is important

to mention that for  $k > 3$  analogs of  $(\tilde{S}_6, \dots, \tilde{S}_{n+2})$  maps  $\mathbf{S} : \mathbb{C}^{(k-2)(n-k)} \mapsto \mathbb{C}^{(k-2)(n-k)}$  may be also constructed [99, 101] and thus the explicit form of  $\Gamma_{k,n+2}^{tree}$  integration contours is known.

So the natural question is how these different expressions can provide us with the representation of the same object? The answer was given in [26, 100]. It turns out, that there actually exists a family of functions  $F^{k=3}(C|t_6, \dots, t_{n+2})$  depending on parameters  $t_6, \dots, t_{n+2}$ , such that:

$$F^{k=3}(C|t_6, \dots, t_{n+2}) = H(C) \frac{1}{S_6(t_6) \dots S_{n+2}(t_{n+2})},$$

$$H(C) = \frac{\prod_{j=6}^{n+1} (12j)(23j-1) \prod_{i=5}^{n+1} (13i)}{(n+1n+21)(123)(234)}, \quad (5.104)$$

with  $(j = 6, \dots, n+2)$

$$S_j = (j-2j-1j)(j12)(23j-2)(j-113) - t_j(j-1j1)(123)(3j-2j-1)(j2j-2), \quad (5.105)$$

so that the result of evaluating by residues at zeros of  $\mathbf{S}(t) = (S_6(t_6), \dots, S_{n+2}(t_{n+2}))$  map the integral (5.96) is  $t_j$  independent [26]:

$$\partial_{t_j} \int_{\mathbf{S}(t)=0} d^{(n-k)} \tau_A F^{k=3}(C|t_6, \dots, t_{n+2})|_{c_{ai} \mapsto c_{ai}(\tau|kin.)} = 0, \text{ for } j = 6, \dots, n+2. \quad (5.106)$$

The case  $t_j = 0$  corresponds to the representation of amplitude obtained from Grassmannian integral representation, while the case  $t_j = 1$  corresponds to the representation obtained from scattering equations representation:

$$F^{k=3}(C|1, \dots, 1) = F^{k=3}(C), \text{ and } F^{k=3}(C|0, \dots, 0) = F_{Gr}^{k=3}(C). \quad (5.107)$$

The obtained relation thus supports the assertion, that Grassmannian integral representation has stringy origin.

As an illustration lets consider simplest case  $k = 3, n + 2 = 6$ . In this case we have integral over single complex parameter  $\tau$  (it is assumed that in all minors the replacement  $c_{ai} \mapsto c_{ai}(\tau|kin.)$  was performed):

$$A_{6,3} = \int_{S(t)=0} d\tau \frac{(135)}{(123)(345)(561)} \frac{1}{S(t)},$$

$$S(t) = t(123)(345)(561)(246) - (234)(456)(612)(351), \quad (5.108)$$

where minors  $(123)$ ,  $(345)$ ,  $(561)$  and  $S(t)$  are liner function of  $\tau$ . According to Cauchy theorem the different residues are related with each other as

$$\{S(t)\} = -\{(123)\} - \{(345)\} - \{(561)\}. \quad (5.109)$$

Here  $\{\dots\}$  denotes the integral residue at the corresponding pole. Note, that for  $(123) = 0$ ,  $(345) = 0$  or  $(561) = 0$  the term in  $S(t)$  proportional to  $t$  vanishes and as a consequence we have

$$\partial_t\{(123)\} = \partial_t\{(345)\} = \partial_t\{(561)\} = 0, \quad (5.110)$$

So, in the computation of the above integral we can put  $S(t)$  to  $S(0)$  and get

$$\frac{(135)}{(123)(345)(561)} \frac{1}{S(0)} = \frac{1}{(123)\dots(612)}. \quad (5.111)$$

In the case  $n + 2 > 6$  the situation is similar, but now one must deal with multiple integrations over complex variables and use global residue theorem [26, 100]. The explicit computations were also preformed for the  $k = 4$  case in [26] and it is believed that one can use  $F(C)$  function in the form of

$$F(C) = F_{Gr}(C) = \frac{1}{(1\dots k)(2\dots k+1)\dots(n+2\dots k-1)}. \quad (5.112)$$

for general values of  $n$  and  $k$ . However, as far as we know there is no general proof of this assertion. Still, throughout this article we assume that for any  $k \geq 3$   $F(C)$  can be chosen as in (5.112).

In previous chapter we used this equivalence for general values of  $n + 2$  and  $k$  to derive our scattering equations representation for  $A_{k,n+1}^*$  amplitude starting from the correlator of vertex operators (string theory amplitude). In the end of this section, as an example let us consider the simple none trivial case of  $n + 2 = 6$ ,  $k = 3$  and check that we indeed get the same result independent of whether we use (5.92) or (5.112) expression for  $F(C)$  function. For residues evaluations it is convenient to introduce the notion of composite residue [22]. For that purpose lets define the set  $S$  of points in  $\mathbb{C}^n$ , such that  $S = \{z | z \in \mathbb{C}^n, s(z) = 0\}$  and  $s(z)$  is some holomorphic function (in our case polynomial). Next, consider  $n$  - form  $\omega = h(z)/s(z)dz$ , where  $dz = dz_1 \wedge \dots \wedge dz_n$  and  $h(z)$  is some other holomorphic function (in our case it is some rational function), and define  $(n - 1)$  - form:

$$res_j[\omega] = (-1)^{j-1} \left( \frac{h(z)}{\partial_{z_j} s(z)} \right) \Big|_S dz_{[j]}, \quad (5.113)$$

with  $dz_{[j]} = dz_1 \wedge \dots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \dots \wedge dz_n$ . Using this definition iteratively we may define  $(n - m)$  - forms as

$$res^m[\omega] = res_m \circ \dots \circ res_1[\omega]. \quad (5.114)$$

These forms are also known as composite residue forms. Computing the string correlation function from the previous section in the case of  $A_{3,4+1}^*$  amplitude we end up with the evaluation of composite residue

$$res_{\beta_1=-1} \circ res_{\beta_2=0}[\omega]. \quad (5.115)$$



Computing the latter provides us with  $F(C)$  function in the form

$$\begin{aligned} F(C) &= \frac{(135)}{(123)(345)(561)} \frac{1}{S}, \\ S &= (123)(345)(561)(245) - (234)(456)(512)(351), \end{aligned} \quad (5.116)$$

which is equivalent in terms of residues evaluation to

$$F(C) = \frac{(612)}{(512)} \frac{1}{(123) \dots (612)}, \quad (5.117)$$

as expected. Also from this example we see, that in the case of  $A_{3,n+1}^*$  off-shell amplitudes we can explicitly construct integration contours for their Grassmannian integral representations (i.e. the maps  $\mathbf{S} = (\tilde{S}_6, \dots, \tilde{S}_{n+2})$ ,  $\mathbf{S} : \mathbb{C}^{(n-3)} \mapsto \mathbb{C}^{(n-3)}$ , whose zeros determine the integration contours  $\Gamma_{3,n+2}^{tree}$  in (5.99)). The latter are given by:

$$\begin{aligned} \tilde{S}_j &= (j - 2j - 1j)(j12)(23j - 2), \quad j = 6, \dots, n + 1, \\ \tilde{S}_{n+2} &= (nn + 1n + 2)(n + 112)(23n). \end{aligned} \quad (5.118)$$

This expression is easily obtained by considering integration contour  $(\tilde{S}_6, \dots, \tilde{S}_{n+2})$  for  $n + 2$  point on-shell amplitude and accounting for  $Reg. \sim (n + 212)/(n + 112)$  factor. It is believed, that in the case  $k > 3$  the integration contours can be constructed in similar fashion.

## 6 Gluing procedure and amplitudes

In this section we would like to discuss the gluing procedure at the level of amplitudes. The latter is equivalent to the evaluation of string correlation function (4.49) with integrals over  $ds_a d\sigma_a$  taken first and integrals over  $d\beta_1 d\beta_2$  evaluated at the end. Here we will restrict ourselves with the case of off-shell amplitudes only. The similar calculations could be of course performed for the case of form factors of operators from stress-tensor operator supermultiplet. To simplify notation it is convenient to introduce gluing operator  $\hat{A}[\dots]$  acting on the space of functions of  $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}_{i=1}^{n+2}$  variables as

$$\hat{A}[f] = \int \prod_{i=n+1}^{n+2} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i d^4 \eta_i}{\text{Vol}[GL(2)]} A_{2,2+1}^* \times f\left(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}_{i=1}^{n+2}\right), \quad (6.119)$$

Performing integration over  $\tilde{\lambda}_{n+1}$ ,  $\tilde{\lambda}_{n+2}$ ,  $\tilde{\eta}_{n+1}$  and  $\tilde{\eta}_{n+2}$  variables as in Appendix A we get

$$\hat{A}[f] = \frac{\langle p\xi \rangle}{\chi^*} \int \frac{d\beta_1}{\beta_1} \frac{d\beta_2}{\beta_2} \frac{1}{\beta_1^2 \beta_2} f\left(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}\right) \Big|_*, \quad (6.120)$$

where  $|_*$  denotes substitutions  $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}_{i=n+1}^{n+2} \mapsto \{\lambda_i(\beta), \tilde{\lambda}_i(\beta), \tilde{\eta}_i(\beta)\}_{i=n+1}^{n+2}$  with

$$\begin{aligned}\lambda_{n+1}(\beta) &= \underline{\lambda}_{n+1} + \beta_2 \underline{\lambda}_{n+2}, & \tilde{\lambda}_{n+1}(\beta) &= \beta_1 \underline{\tilde{\lambda}}_{n+1} + \frac{(1+\beta_1)}{\beta_2} \underline{\tilde{\lambda}}_{n+2}, & \tilde{\eta}_{n+1}(\beta) &= -\beta_1 \underline{\tilde{\eta}}_{n+1}, \\ \lambda_{n+2}(\beta) &= \underline{\lambda}_{n+2} + \frac{(1+\beta_1)}{\beta_1 \beta_2} \underline{\lambda}_{n+1}, & \tilde{\lambda}_{n+2}(\beta) &= -\beta_1 \underline{\tilde{\lambda}}_{n+2} - \beta_1 \beta_2 \underline{\tilde{\lambda}}_{n+1}, & \tilde{\eta}_{n+2}(\beta) &= \beta_1 \beta_2 \underline{\tilde{\eta}}_{n+1}.\end{aligned}\tag{6.121}$$

and

$$\underline{\lambda}_{n+1} = \lambda_p, \quad \underline{\tilde{\lambda}}_{n+1} = \frac{\langle \xi | k}{\langle \xi p \rangle}, \quad \underline{\tilde{\eta}}_n = \tilde{\eta}_p; \quad \underline{\lambda}_{n+2} = \lambda_\xi, \quad \underline{\tilde{\lambda}}_{n+2} = \frac{\langle p | k}{\langle \xi p \rangle}, \quad \underline{\tilde{\eta}}_{n+2} = 0. \tag{6.122}$$

The transformed momenta for  $n+1$ 'th and  $n+2$ 'th particles are then given by

$$\begin{aligned}p_{n+1}(\beta) &= -\beta_1 \underline{\lambda}_{n+1} \underline{\tilde{\lambda}}_{n+1} + \frac{1+\beta_1}{\beta_2} \underline{\lambda}_{n+1} \underline{\tilde{\lambda}}_{n+2} - \beta_1 \beta_2 \underline{\lambda}_{n+2} \underline{\tilde{\lambda}}_{n+1} + (1+\beta_1) \underline{\lambda}_{n+2} \underline{\tilde{\lambda}}_{n+2}, \\ p_{n+2}(\beta) &= -\beta_1 \underline{\lambda}_{n+2} \underline{\tilde{\lambda}}_{n+2} - \frac{1+\beta_1}{\beta_2} \underline{\lambda}_{n+1} \underline{\tilde{\lambda}}_{n+2} + \beta_1 \beta_2 \underline{\lambda}_{n+2} \underline{\tilde{\lambda}}_{n+1} + (1+\beta_1) \underline{\lambda}_{n+1} \underline{\tilde{\lambda}}_{n+1}.\end{aligned}\tag{6.123}$$

and using definitions above it is easy to see that

$$\begin{aligned}k &= \lambda_{n+1}(\beta) \tilde{\lambda}_{n+1}(\beta) + \lambda_{n+2}(\beta) \tilde{\lambda}_{n+2}(\beta), \\ \lambda_p \eta_p &= \lambda_{n+1}(\beta) \tilde{\eta}_{n+1}(\beta) + \lambda_{n+2}(\beta) \tilde{\eta}_{n+2}(\beta),\end{aligned}\tag{6.124}$$

for all values of  $\beta_1$  and  $\beta_2$ .

Now let's see how our gluing procedure works on some explicit examples. The simplest example is given by  $k=2$   $n+2=4$  point amplitude  $A_{2,4}$ . Note, that next steps are actually identical for all  $k=2$  amplitudes with arbitrary  $n$ . In the case of  $A_{2,4}$  amplitude we have

$$A_{2,4} = \delta^4(p_{1234}) \frac{\delta^8(q_{1234})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.\tag{6.125}$$

Introducing notations<sup>21</sup>

$$p_{1\dots n} \equiv \sum_{i=1}^n p_i \equiv \sum_{i=1}^n \lambda_i \tilde{\lambda}_i, \quad p_{1\dots n}^2 = p_{1,n}^2, \quad q_{1\dots n} \equiv \sum_{i=1}^n \lambda_i \tilde{\eta}_i.\tag{6.126}$$

the  $A_{2,4}$  amplitude with  $|_*$  substitution applied takes the form

$$A_{2,4}|_* = \delta^4(p_{12} + k) \frac{\delta^8(q_{12p}) \beta_1^2 \beta_2}{\langle 12 \rangle (\langle 2p \rangle + \beta_2 \langle 2\xi \rangle) \langle p\xi \rangle (\beta_1 \beta_2 \langle 1\xi \rangle + (1+\beta_1) \langle 1p \rangle)}.\tag{6.127}$$

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<sup>21</sup>Here for simplicity we also drop spinorial and  $SU(4)_R$  indices.

Now, evaluating integral over  $\beta_1, \beta_2$  by means of composite residue  $res_{\beta_1=-1} \circ res_{\beta_2=0}[\dots]$  we get

$$\hat{A}[A_{2,4}] = \delta^4(p_{12} + k) \frac{\delta^8(q_{12p})}{\kappa^* \langle 12 \rangle} res_{\beta_2=0} \circ res_{\beta_1=-1}[\omega] = A_{2,2+1}^*, \quad (6.128)$$

where

$$\omega = \frac{d\beta_2 \wedge d\beta_1}{\beta_2 \beta_1 (\langle 2p \rangle + \beta_2 \langle 2\xi \rangle) (\beta_1 \beta_2 \langle 1\xi \rangle + (1 + \beta_1) \langle 1p \rangle)}. \quad (6.129)$$

The similar result<sup>22</sup> will obviously hold for  $A_{2,n+2}$  amplitude, that is

$$\hat{A}[A_{2,n+2}] = A_{2,n+1}^*. \quad (6.130)$$

Note also that the gluing operation commutes with projectors on particular physical particles provided we identify  $n+1$  and  $n+2$  particles with gluons with  $-+$  polarizations. Indeed from the previous example we have [51, 70]

$$\partial_{\tilde{\eta}_2}^4 \partial_{\tilde{\eta}_p}^4 \hat{A}[A_{2,4}] = \hat{A}[\partial_{\tilde{\eta}_2}^4 \partial_{\tilde{\eta}_3}^4 A_{2,4}] = \frac{\delta^4(p_{12} + k)}{\kappa^* \langle 12 \rangle} res_{\beta_1=-1} \circ res_{\beta_2=0}[\omega] = A_{2,2+1}^*(1^+ 2^- g_3^*), \quad (6.131)$$

where  $\omega$  is given now by

$$\omega = \frac{(\langle 2p \rangle + \beta_2 \langle 2\xi \rangle)^3 d\beta_2 \wedge d\beta_1}{\beta_2 \beta_1 (\beta_1 \beta_2 \langle 1\xi \rangle + (1 + \beta_1) \langle 1p \rangle)}. \quad (6.132)$$

Lets proceed with more involved examples and reproduce results for  $A_{3,3+1}^*(1^- 2^- 3^+ g_4^*)$  and  $A_{3,4+1}^*(1^+ 2^+ 3^- 4^- g_5^*)$  amplitudes from [51]. In the case of  $A_{3,3+1}^*(1^- 2^- 3^+ g_4^*)$  amplitude we have to start with  $A_{3,5}(1^- 2^- 3^+ 4^- 5^+)$  amplitude (here and below  $c^{-1} = \langle p\xi \rangle$ ):

$$\partial_{\tilde{\eta}_1}^4 \partial_{\tilde{\eta}_2}^4 \partial_{\tilde{\eta}_4}^4 A_{3,5} = A_{3,5}(1^- 2^- 3^+ 4^- 5^+) = \delta^4(p_{12345}) \frac{[35]^4}{[12][23][34][45][51]}, \quad (6.133)$$

so that

$$A_{3,5}(1^- 2^- 3^+ 4^- 5^+) \Big|_* = \frac{\delta^4(p_{123} + k) \beta_1^2 \beta_2 (\kappa^* c^{-1} [p3] + \beta_2 [3\underline{4}])^4}{k^2 c^{-1} [12][23] (-\beta_1 \beta_2 [3\underline{4}] + (1 + \beta_1) c^{-1} \kappa^* [3p]) ([1p] c^{-1} \kappa^* + \beta_2 [1\underline{4}])}. \quad (6.134)$$

Now, recalling that  $k^2 = -\kappa^* \kappa$  we get [51]

$$\hat{A}[\partial_{\tilde{\eta}_1}^4 \partial_{\tilde{\eta}_2}^4 \partial_{\tilde{\eta}_4}^4 A_{3,5}] = \frac{\delta^4(p_{123} + k) [p3]^3}{\kappa [12][23][p1]} = A_{2,3+1}^*(1^- 2^- 3^+ g_4^*), \quad (6.135)$$

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<sup>22</sup>It can be obtained by simple spinor relabellings from previous example

and the integration with respect to  $\beta$ 's was performed by taking composite residue  $res_{\beta_1=-1} \circ res_{\beta_2=0}[\dots]$ .

In a similar fashion for  $A_{3,5}(1^+2^+3^-4^-5^-6^+)$  amplitude we have

$$\partial_{\tilde{\eta}_3}^4 \partial_{\tilde{\eta}_4}^4 \partial_{\tilde{\eta}_5}^4 A_{3,5} = A_{3,5}(1^+2^+3^-4^-5^-6^+) = A + B, \quad (6.136)$$

with

$$\begin{aligned} A &= \frac{\langle 3|1+2|6]^3}{[45][56]} \frac{\delta^4(p_{1\dots 6})}{\langle 12 \rangle \langle 23 \rangle p_{1,3}^2 \langle 1|2+3|4]}, \\ B &= \frac{\langle 5|3+4|2]^3}{\langle 56 \rangle \langle 61 \rangle} \frac{\delta^4(p_{1\dots 6})}{[23][34] p_{2,4}^2 \langle 1|2+3|4]}. \end{aligned} \quad (6.137)$$

Next, it is not hard to see that  $([x] \equiv \langle 3|(1+2), |y] \equiv (3+4)|2])$

$$\begin{aligned} A|_* &= \frac{\beta_1^2 \beta_2 (\langle py \rangle + \beta_2 \langle \xi y \rangle)^3}{c (\langle 1\xi \rangle \beta_1 \beta_2 + (1+\beta_1) \langle p1 \rangle)} \frac{\delta^4(p_{1234} + k)}{\langle 12 \rangle \langle 23 \rangle p_{1,3}^2 \langle 1|2+3|4]}, \\ B|_* &= \frac{\beta_1^2 \beta_2 ([px] c^{-1} \kappa^* + \beta_2 [\underline{5}x])^3}{c^{-1} \kappa \kappa^* (\beta_1^2 \beta_2 [4\underline{5}] + (1+\beta_1) [4p] \kappa^* c^{-1})} \frac{\delta^4(p_{1234} + k)}{[23][34] p_{2,4}^2 \langle 1|2+3|4]}. \end{aligned} \quad (6.138)$$

Now, defining

$$\begin{aligned} \omega_A &= \frac{([px] c^{-1} \kappa^* + \beta_2 [\underline{5}x])^3}{c^{-1} \kappa \kappa^* (\beta_1 \beta_2 [4\underline{5}] + (1+\beta_1) [4p] \kappa^* c^{-1})} \frac{d\beta_2 \wedge d\beta_1}{\beta_1 \beta_2}, \\ \omega_B &= \frac{(\langle py \rangle + \beta_2 \langle \xi y \rangle)^3}{c (\langle 1\xi \rangle \beta_1 \beta_2 + (1+\beta_1) \langle p1 \rangle)} \frac{d\beta_2 \wedge d\beta_1}{\beta_1 \beta_2}, \end{aligned} \quad (6.139)$$

we get

$$\hat{A}[A] = \frac{\delta^4(p_{1234} + k)}{\langle 12 \rangle \langle 23 \rangle p_{1,3}^2 \langle 1|2+3|4]} \frac{1}{c^{-1} \kappa^*} res_{\beta_1=-1} \circ res_{\beta_2=0}[\omega_A] = \frac{1}{\kappa} \frac{\delta^4(p_{1234} + k) \langle 3|1+2|p]^3}{\langle 12 \rangle \langle 23 \rangle [4p] p_{1,3}^2 \langle 1|2+3|4]}, \quad (6.140)$$

and

$$\hat{A}[B] = \frac{\delta^4(p_{1234} + k)}{[23][34] p_{2,4}^2 \langle 1|2+3|4]} \frac{1}{c^{-1} \kappa^*} res_{\beta_1=-1} \circ res_{\beta_2=0}[\omega_B] = \frac{1}{\kappa^*} \frac{\delta^4(p_{1234} + k) \langle p|3+4|2]^3}{\langle p1 \rangle [23][34] p_{2,4}^2 \langle 1|2+3|4]}, \quad (6.141)$$

So, as expected [51]

$$\hat{A}[A_{3,5}(1^+2^+3^-4^-5^-6^+)] = A_{3,4+1}^*(1^+2^+3^-4^-g_5^*). \quad (6.142)$$

As a final example we would like to consider the case with multiple gluing operations applied. That is we are considering the situation with multiple Wilson operator  $\mathcal{V}_{i,i+1}^{\text{WL}}$  insertions. Lets consider quite none trivial example of

$$A_{3,0+3}^*(g_1^*, g_2^*, g_3^*) = \langle \mathcal{V}_{1,2}^{\text{WL}} \mathcal{V}_{3,4}^{\text{WL}} \mathcal{V}_{5,6}^{\text{WL}} \rangle. \quad (6.143)$$

To distinguish gluing operation with respect to different legs we will explicitly write the legs numbers on which  $\hat{A}$  operator acts. For example,  $\hat{A}_{12}$  denotes gluing operation with respect to legs 1 and 2. According to our previous discussion  $A_{3,0+3}^*$  amplitude could be written as

$$A_{3,0+3}^*(g_1^*, g_2^*, g_3^*) = (\hat{A}_{12} \circ \hat{A}_{34} \circ \hat{A}_{56})[A_{3,6}(1^- 2^+ 3^- 4^+ 5^- 6^+)],, \quad (6.144)$$

where  $A_{3,6}(1^- 2^+ 3^- 4^+ 5^- 6^+)$  amplitude is given by

$$A_{3,6} = \delta^4(p_{1\dots 6}) (1 + \mathbb{P}^2 + \mathbb{P}^4) f, \quad f = \frac{\langle 13 \rangle^4 [46]^4}{\langle 12 \rangle \langle 23 \rangle [45] [56] \langle 3|1+2|6 \rangle \langle 1|5+6|4 \rangle p_{456}^2} \quad (6.145)$$

and  $\mathbb{P}$  is permutation operator shifting spinor labels by  $+1 \bmod 6$ . The algebraic manipulation related to the actions of  $\hat{A}_{ii+1}$  operators are identical to those already discussed. The factors  $1/\beta_1^2 \beta_2$  in the definition of gluing operators will cancel with corresponding factors in the amplitude after substitutions applied, while integrals are evaluated by composite residues  $\text{res}_{\beta_1=-1} \circ \text{res}_{\beta_2=0}$ . So, in what follows we will present only the results of applying gluing operators  $\hat{A}_{ii+1}$  to on-shell amplitude. For  $f$  term we have:

$$\hat{A}_{56}[f] = \delta^4(p_{1234} + k_3) \frac{\langle 13 \rangle^4 [4p_3]^3}{\kappa_3 \langle 12 \rangle \langle 23 \rangle \langle 3|1+2|p_3 \rangle \langle 1|k_3|4 \rangle p_{123}^2}, \quad (6.146)$$

and

$$(\hat{A}_{34} \circ \hat{A}_{56})[f] = \delta^4(p_{12} + k_2 + k_3) \frac{\langle 1p_2 \rangle^4 [p_2 p_3]^3}{\kappa_3 \langle 12 \rangle \langle 2p_2 \rangle \langle p_2|1+2|p_3 \rangle \langle 1|k_3|p_2 \rangle \langle p_2|k_3|p_2 \rangle}. \quad (6.147)$$

Note that the ordinary propagator  $1/p_{123}^2$  transformed into eikonal one  $1/(p_2 k_3)$  after the action of gluing operator. Finally

$$(\hat{A}_{12} \circ \hat{A}_{34} \circ \hat{A}_{56})[f] = \delta^4(k_1 + k_2 + k_3) \frac{\langle p_1 p_2 \rangle^3 [p_2 p_3]^3}{\kappa_3 \kappa_1^* \langle p_2|k_1|p_3 \rangle \langle p_1|k_3|p_2 \rangle \langle p_2|k_1|p_2 \rangle}, \quad (6.148)$$

where we used that  $\langle p_2|k_3|p_2 \rangle = \langle p_2|k_1|p_2 \rangle$ . Other terms can be obtained by similar computations or just by careful relabeling of indexes. The final result takes the form:

$$\begin{aligned} A_{3,0+3}^* &= (\hat{A}_{12} \circ \hat{A}_{34} \circ \hat{A}_{56})[A_{3,6}(1^- 2^+ 3^- 4^+ 5^- 6^+)] = \delta^4(k_1 + k_2 + k_3) (1 + \mathbb{P}' + \mathbb{P}'^2) \tilde{f}, \\ \tilde{f} &= \frac{\langle p_1 p_2 \rangle^3 [p_2 p_3]^3}{\kappa_3 \kappa_1^* \langle p_2|k_1|p_3 \rangle \langle p_1|k_3|p_2 \rangle \langle p_2|k_1|p_2 \rangle}. \end{aligned} \quad (6.149)$$

Here  $\mathbb{P}'$  is permutation operator which now shifts all spinor and momenta labels by  $+1 \bmod 3$ . Obtained expression is in full agreement with previous computations using both Grassmannian integral representation [52] and BCFW recursion [70].

At the end we would like to point out the formal analogy between the action of  $\hat{A}_{ii+1}$  operators on  $A_{k,n}$  amplitudes and action of  $R$ -matrices on some vacuum state of integrable spin chain. Indeed, it looks like  $\hat{A}_{ii+1}$  operator creates "excitation" (Wilson-line operator insertion) in the "vacuum" consisting from on-shell states. We think that this analogy being properly investigated may provide us with the answer to question "what is the appropriate description of Wilson line form factors in terms of some integrable system?".

It is also interesting to note that the integration with respect to  $\beta_1, \beta_2$  variables, which was performed by taking residues, is in fact equivalent to the choice of specific kinematical limit for the momenta  $p_{n+1}$  and  $p_{n+2}$  of the initial on-shell amplitude. If we naively take consecutive limits  $\beta_1 \rightarrow -1, \beta_2 \rightarrow 0$  in the definitions of momenta  $p_{n+1}(\beta)$  and  $p_{n+2}(\beta)$ , which is equivalent to residue evaluation, we would get finite result<sup>23</sup>

$$p_{n+1} = \underline{\lambda}_{n+1} \tilde{\underline{\lambda}}_{n+1}, \quad p_{n+2} = \underline{\lambda}_{n+2} \tilde{\underline{\lambda}}_{n+2}. \quad (6.150)$$

On the other hand, for all  $\omega$  forms which we have encountered in previous examples we may use global residue theorem to relate multiple residue at  $\beta_1 = -1, \beta_2 = 0$  with the multiple residue at  $\beta_1 = 0, \beta_2 = 0$ . If we take the limits  $\beta_1 \rightarrow 0, \beta_2 \rightarrow 0$  (regardless of the order of limits) in the definitions of  $p_{n+1}(\beta)$  and  $p_{n+2}(\beta)$  momenta we will get singular result

$$\begin{aligned} p_{n+1} &= \frac{1}{\beta_2} \underline{\lambda}_{n+1} \tilde{\underline{\lambda}}_{n+2} + \underline{\lambda}_{n+2} \tilde{\underline{\lambda}}_{n+2} + O(\beta_2), \\ p_{n+2} &= \frac{1}{\beta_2} \underline{\lambda}_{n+1} \tilde{\underline{\lambda}}_{n+2} + \underline{\lambda}_{n+1} \tilde{\underline{\lambda}}_{n+1} + O(\beta_2), \end{aligned} \quad (6.151)$$

which is equivalent to BCFW shift  $[n+1, n+2\rangle$  of  $p_{n+1} = \underline{\lambda}_{n+2} \tilde{\underline{\lambda}}_{n+2}$  and  $p_{n+2} = \underline{\lambda}_{n+1} \tilde{\underline{\lambda}}_{n+1}$  momenta evaluated at large  $z$ . The behavior of amplitudes in the limit  $z \rightarrow \infty$  may be interpreted as special kinematical limit with some particles with large (complex) light like momenta traveling in the soft background [102]. So in this sense our gluing procedure is closely related to the specific high energy kinematical limit of the ordinary on-shell amplitudes.

## 7 Conclusion

In this paper we presented results for scattering equations representations for form factors of local and Wilson line operators in  $\mathcal{N} = 4$  SYM derived from corresponding four

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<sup>23</sup>Note, that if we would take the limits in opposite order the results for  $p_{n+1}, p_{n+2}$  momenta would diverge, but the expression for off-shell amplitudes would still be finite.

dimensional ambitwistor string theory. In the case of local operators we restricted ourselves to the case of form factors of operators from stress-tensor operator supermultiplet. The obtained results are in agreement with previously obtained Grassmannian integral representations. As by product we discovered an easy and convenient gluing procedure, which allows to obtain required form factor expressions from already known amplitude expressions. The construction of composite string vertex operators for the analysed local or Wilson line operators was inspired by the mentioned gluing procedure. An interesting future research direction, which we are planning to pursue, will be to consider pullbacks of composite operators defined on twistor or Lorentz harmonic chiral superspace [54–60]. We hope that our consideration along these lines could be extended to arbitrary local composite operators.

Next, it would be very interesting to fully uncover geometrical picture behind Grassmannian and scattering equations representations for form factors of local and Wilson line operators. It is interesting to see if the “Amplituhedron” picture could be extended to all possible gauge invariant observables in  $\mathcal{N} = 4$  SYM, which representations under global gauge transformations may differ from singlet representation.

Having obtained scattering equations representations one may wonder what is the most efficient way to get final expressions for particular form factors with given number of particles and their helicities. In the case of amplitudes we know that it is given by computation of global residues with the methods of computational algebraic geometry [103–105], see also [101]. It would be interesting to see how this procedure works in the case of tree level form factors, their loop level integrands and provide necessary details needed when writing computer codes.

Finally, it would be interesting to consider loop corrections to form factors of Wilson line operators (gauge invariant off-shell amplitudes). Also, it is extremely interesting to see how the presented approach works in other theories, for example in gravity and supergravity, where in the case of gauge invariant off-shell amplitudes we have a well developed framework based on high-energy effective lagrangian [106–110], see also [111–115] for similar research along this direction.

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## A Gluing procedure and Grassmannians

The gluing procedure introduced in [48] was already used to obtain both the Grassmannian integral representation for form factors of operators from stress-tensor supermultiplet [48] and off-shell amplitudes with one leg off-shell [51]. The idea is to take a top-cell diagram for amplitude, perform a sequence of square and merge/unmerge moves until we get a box on the boundary and replace it with the corresponding minimal form factor or off-shell amplitude. Graphically, this relation reads<sup>24</sup>

(A.152)

where the box at the legs  $n + 1$  and  $n + 2$  was replaced for the sake of concreteness. We got a similar relation of form factor on-shell diagrams to the amplitude on-shell diagrams in [47, 50] based on soft limit procedure. The corresponding box diagram was deformed by extra soft factor, so that it became equivalent to the corresponding minimal form factor.

It turns out however, that there is a simpler gluing procedure. Namely, we can glue (perform the on-shell phase space integration) minimal form factor or off-shell amplitude directly to the amplitude top cell diagram without cutting off mentioned above boxes. The motivation behind this construction is given by the fact, that by gluing (this time introducing loop integration) for example the minimal form factor to on-shell amplitude we get one-loop form factor, whose leading singularity (extracted by maximally cutting the loop) gives us the corresponding tree level form factors. Below and in the next Appendix we will see how this new gluing procedure works in details.

To begin with, let's start with the Grassmannian integral representation for the amplitude with one leg off-shell. According to our new gluing procedure we have<sup>25</sup>

$$A_{k,n+1}^* = \int \prod_{i=n+1}^{n+2} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i}{\text{Vol}[GL(1)]} d^4 \tilde{\eta}_i A_{2,2+1}^*(g^*, \Omega_{n+1}, \Omega_{n+2}) \Big|_{\lambda \rightarrow -\lambda} A_{k,n+2}, \quad (\text{A.153})$$

where  $A_{k,n+2}$  is  $N^{k-2}$ MHV  $n + 2$  point on-shell amplitude<sup>26</sup>:

$$A_{k,n+2} = \int \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \frac{\delta^{k \times 2}(C \cdot \tilde{\lambda}) \delta^{k \times 4}(C \cdot \tilde{\eta}) \delta^{(n+2-k) \times 2}(C^\perp \cdot \lambda)}{(1 \cdots k)(2 \cdots k+1) \cdots (n+2 \cdots k-1)}, \quad (\text{A.154})$$

<sup>24</sup>We have borrowed this nice picture from [48]

<sup>25</sup>Without loss of generality we may choose off-shell leg to lie between legs 1 and  $n$ .

<sup>26</sup>Here we left the integration contour unspecified.



and the minimal off-shell vertex  $A_{2,2+1}^*(g^*, n+1, n+2)$  is given by [51]:

$$\begin{aligned} A_{2,2+1}^*(g^*, \Omega_{n+1}, \Omega_{n+2}) &= \frac{1}{\kappa^*} \int \frac{d^{2 \times 3} C}{\text{Vol}[GL(2)]} \frac{\delta^4(C \cdot \tilde{\lambda}) \delta^8(C \cdot \tilde{\eta}) \delta^4(C^\perp \cdot \lambda)}{(pn+1)(n+1)(n+2)(n+2p)} \\ &= \frac{1}{\kappa^*} \int \frac{d\beta_1}{\beta_1} \frac{d\beta_2}{\beta_2} \delta^2(\lambda_p + \beta_1 \lambda_{n+1} - \beta_1 \beta_2 \lambda_{n+2}) \delta^2(\tilde{\lambda}_{n+1}) \delta^2(\tilde{\lambda}_{n+2}) \\ &\quad \times \delta^4(\tilde{\eta}_{n+1} + \beta_1 \tilde{\eta}_p) \delta^4(\tilde{\eta}_{n+2} + \beta_2 \tilde{\eta}_{n+1}) \end{aligned} \quad (\text{A.155})$$

with  $\tilde{\lambda}_{n+1} = \tilde{\lambda}_{n+1} + \frac{\langle n+2|k}{\langle n+2|n+1\rangle}$  and  $\tilde{\lambda}_{n+2} = \tilde{\lambda}_{n+2} + \frac{\langle n+1|k}{\langle n+1|n+2\rangle}$ . Here  $p$  is the off-shell gluon direction and  $k$  is its momentum. Now, the integration steps up to final integrations in  $\beta_1$  and  $\beta_2$  follow closely those in [51]. That is, performing integrations over  $\tilde{\lambda}_{n+1}$ ,  $\tilde{\lambda}_{n+2}$ ,  $\tilde{\eta}_{n+1}$  and  $\tilde{\eta}_{n+2}$  we get

$$\tilde{\lambda}_{n+1} = -\frac{\langle n+2|k}{\langle n+2|n+1\rangle}, \quad \tilde{\lambda}_{n+2} = -\frac{\langle n+1|k}{\langle n+1|n+2\rangle}, \quad (\text{A.156})$$

$$\tilde{\eta}_{n+1} = -\beta_1 \tilde{\eta}_p, \quad \tilde{\eta}_{n+2} = \beta_1 \beta_2 \tilde{\eta}_p. \quad (\text{A.157})$$

The  $\text{Vol}[GL(1)]^2$  redundancy in the remaining integrations over  $\lambda$  is removed using their parametrization as in [48]:

$$\lambda_{n+1} = \xi_A - \beta_3 \xi_B, \quad \lambda_{n+2} = \xi_B - \beta_4 \xi_A, \quad (\text{A.158})$$

where  $\xi_A$  and  $\xi_B$  are two arbitrary but linearly independent reference spinors. Then  $\langle n+1 \ n+2 \rangle = (\beta_3 \beta_4 - 1) \langle \xi_B \xi_A \rangle$ ,

$$\int \frac{d^2 \lambda_{n+1}}{\text{Vol}[GL(1)]} \frac{d^2 \lambda_{n+2}}{\text{Vol}[GL(1)]} = -\langle \xi_A \xi_B \rangle^2 \int d\beta_3 d\beta_4 \quad (\text{A.159})$$

and

$$\begin{aligned} A_{k,n+1}^* &= \frac{1}{\kappa^*} \langle \xi_A \xi_B \rangle^2 \int \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \frac{d\beta_1}{\beta_1} \frac{d\beta_2}{\beta_2} \frac{d\beta_3 d\beta_4}{(1 - \beta_3 \beta_4)^2} \\ &\quad \times \delta^2(\lambda_p + \beta_1(1 + \beta_2 \beta_4) \xi_A - \beta_1(\beta_2 + \beta_3) \xi_B) \\ &\quad \times \frac{1}{(1 \cdots k) \cdots (n+2 \cdots k-1)} \delta^{k \times 2}(C' \cdot \tilde{\underline{\lambda}}) \delta^{k \times 4}(C' \cdot \tilde{\underline{\eta}}) \delta^{(n+2-k) \times 2}(C'^\perp \cdot \underline{\lambda}). \end{aligned} \quad (\text{A.160})$$

Here, we introduced the following notation

$$\begin{aligned} C'_{n+1} &= \frac{1}{1 - \beta_3 \beta_4} C_{n+1} + \frac{\beta_3}{1 - \beta_3 \beta_4} C_{n+2}, & C'^\perp_{n+1} &= C_{n+1}^\perp - \beta_4 C_{n+2}^\perp, \\ C'_{n+2} &= \frac{1}{1 - \beta_3 \beta_4} C_{n+2} + \frac{\beta_4}{1 - \beta_3 \beta_4} C_{n+1}, & C'^\perp_{n+2} &= C_{n+2}^\perp - \beta_3 C_{n+1}^\perp, \end{aligned} \quad (\text{A.161})$$

and

$$\begin{aligned}
\underline{\underline{\lambda}}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\underline{\lambda}}_{n+1} &= \xi_A, & \underline{\underline{\lambda}}_{n+2} &= \xi_B \\
\underline{\underline{\tilde{\lambda}}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\underline{\tilde{\lambda}}}_{n+1} &= \frac{\langle \xi_B | k \rangle}{\langle \xi_B \xi_A \rangle}, & \underline{\underline{\tilde{\lambda}}}_{n+2} &= -\frac{\langle \xi_A | k \rangle}{\langle \xi_B \xi_A \rangle}, \\
\underline{\underline{\tilde{\eta}}}_i &= \tilde{\eta}_i, & i &= 1, \dots, n, & \underline{\underline{\tilde{\eta}}}_{n+1} &= \tilde{\eta}_p, & \underline{\underline{\tilde{\eta}}}_{n+2} &= 0.
\end{aligned} \tag{A.162}$$

The factor of  $1/(1 - \beta_3\beta_4)^2$  in (A.160) is the Jacobian from reorganizing the  $C^\perp \cdot \lambda$  delta functions (see [48] for further details). Next, rewriting the first delta function in (A.160) as

$$\begin{aligned}
&\delta^2(\lambda_p + \beta_1(1 + \beta_2\beta_4)\xi_A - \beta_1(\beta_2 + \beta_3)\xi_B) \\
&= \frac{1}{\beta_1^2\beta_2\langle \xi_A \xi_B \rangle} \delta(\beta_3 - \frac{\langle \xi_{AP} \rangle}{\beta_1\langle \xi_A \xi_B \rangle} + \beta_2) \cdot \delta(\beta_4 - \frac{\langle \xi_{BP} \rangle}{\beta_1\beta_2\langle \xi_A \xi_B \rangle} + \frac{1}{\beta_2}) \tag{A.163}
\end{aligned}$$

choosing  $\xi_A = \lambda_p$ ,  $\xi_B = \xi$  and taking integrations over  $\beta_3, \beta_4$  we get

$$A_{k,n+1}^* = \frac{1}{\kappa^*} \langle \xi p \rangle \int \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \frac{d\beta_1 d\beta_2}{\beta_1 \beta_2^2} \frac{\delta^{k \times 2}(C' \cdot \underline{\underline{\tilde{\lambda}}}) \delta^{k \times 4}(C' \cdot \underline{\underline{\tilde{\eta}}}) \delta^{(n+2-k) \times 2}(C'^\perp \cdot \underline{\underline{\lambda}})}{(1 \cdots k)(2 \cdots k+1) \cdots (n+2 \cdots k-1)}, \tag{A.164}$$

where now

$$\begin{aligned}
C'_{n+1} &= -\beta_1 C_{n+1} + \beta_1 \beta_2 C_{n+2}, & C'^\perp_{n+1} &= C^\perp_{n+1} + \frac{1 + \beta_1}{\beta_1 \beta_2} C^\perp_{n+2}, \\
C'_{n+2} &= -\beta_1 C_{n+2} + \frac{1 + \beta_1}{\beta_2} C_{n+1}, & C'^\perp_{n+2} &= C^\perp_{n+2} + \beta_2 C^\perp_{n+1},
\end{aligned} \tag{A.165}$$

and

$$\begin{aligned}
\underline{\underline{\lambda}}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\underline{\lambda}}_{n+1} &= \lambda_p, & \underline{\underline{\lambda}}_{n+2} &= \xi \\
\underline{\underline{\tilde{\lambda}}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\underline{\tilde{\lambda}}}_{n+1} &= \frac{\langle \xi | k \rangle}{\langle \xi p \rangle}, & \underline{\underline{\tilde{\lambda}}}_{n+2} &= -\frac{\langle p | k \rangle}{\langle \xi p \rangle}, \\
\underline{\underline{\tilde{\eta}}}_i &= \tilde{\eta}_i, & i &= 1, \dots, n, & \underline{\underline{\tilde{\eta}}}_{n+1} &= \tilde{\eta}_p, & \underline{\underline{\tilde{\eta}}}_{n+2} &= 0.
\end{aligned} \tag{A.166}$$

Introducing inverse  $C$ -matrix transformation

$$\begin{aligned}
C_{n+1} &= C'_{n+1} + \beta_2 C'_{n+2} \\
C_{n+2} &= \frac{1 + \beta_1}{\beta_1 \beta_2} C'_{n+1} + C'_{n+2}
\end{aligned} \tag{A.167}$$

minors of  $C$ -matrix containing both  $n+1$  and  $n+2$  columns when rewritten in terms of minors of  $C'$ -matrix acquire extra  $-\frac{1}{\beta_1}$  factor. For example, for  $(n+1 \cdots k-2)$  minor we have

$$(n+1 \cdots k-2) = -\frac{1}{\beta_1}(n+1 \cdots k-2)'. \quad (\text{A.168})$$

Minors containing either  $n+1$  or  $n+2$  column transform as

$$(n+2 \ 1 \cdots k-1) = \frac{1+\beta_1}{\beta_1\beta_2}(n+1 \ 1 \cdots k-1)' + (n+2 \ 1 \cdots k-1)', \quad (\text{A.169})$$

$$(n-k+2 \cdots n+1) = (n-k+2 \cdots n+1)' + \beta_2(n-k+2 \cdots n+2)', \quad (\text{A.170})$$

while all other minors remain unchanged  $(\cdots) = (\cdots)'$ . Finally, accounting for the Jacobian of transformation  $\left(-\frac{1}{\beta_1}\right)^k$  we get

$$\begin{aligned} A_{k,n+1}^* &= -\frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d^{k \times (n+2)} C'}{\text{Vol}[GL(k)]} \frac{d\beta_1 d\beta_2}{\beta_1 \beta_2} \delta^{k \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp} \cdot \underline{\lambda}) \\ &\times \frac{1}{(1 \cdots k)' \cdots (n+2 \cdots k-1)' \left(1 + \beta_2 \frac{(n-k+2 \cdots n+2)'}{(n-k+2 \cdots n+1)'}\right) \left(\beta_1 \beta_2 + (1 + \beta_1) \frac{(n+1 \cdots k-1)}{(n+2 \cdots k-1)}\right)} \end{aligned}$$

Now, taking first the residue at  $\beta_2 = 0$  and then at  $\beta_1 = -1$  we recover our result from [51]:

$$A_{k,n+1}^* = \int_{\Gamma_{k,n+2}^{tree}} \frac{d^{k \times (n+2)} C'}{\text{Vol}[GL(k)]} \text{Reg.} \frac{\delta^{k \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp} \cdot \underline{\lambda})}{(1 \cdots k)' \cdots (n+1 \cdots k-2)' (n+2 \ 1 \cdots k-1)',} \quad (\text{A.171})$$

with

$$\text{Reg.} = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(n+2 \ 1 \cdots k-1)'}{(n+1 \ 1 \cdots k-1)'}. \quad (\text{A.172})$$

As we already mentioned this new gluing procedure could be also applied to the case of form factors of operators from stress-tensor operator supermultiplet. In this case we have

$$F_{k,n} = \int \prod_{i=n+1}^{n+2} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i}{\text{Vol}[GL(1)]} d^4 \tilde{\eta}_i F_{2,2}(\Omega_{n+1}, \Omega_{n+2}; \mathcal{T}) \Big|_{\lambda \rightarrow -\lambda} A_{k,n+2} + \text{other gluing positions.}, \quad (\text{A.173})$$

where the minimal form factor  $F_{2,2}(\Omega_{n+1}, \Omega_{n+2}; \mathcal{T})$  is given by (4.71). Performing next on-shell integrations for particles  $n+1$  and  $n+2$  as above<sup>27</sup> we get (Here we are considering

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<sup>27</sup>See [48] for details.

only single term, corresponding to the gluing of minimal form factor between legs 1 and  $n$ . Other terms come from gluing between legs  $i$  and  $i + 1$ ,  $i = 1 \dots n - 1$ )

$$F_{k,n} = -\langle \xi_A \xi_B \rangle^2 \int \frac{d\beta_1 d\beta_2}{(1 - \beta_1 \beta_2)^2} \int \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \frac{\delta^{k \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp} \cdot \underline{\lambda})}{(1 \dots k)(2 \dots k+1) \dots (n+2 \dots k-1)}, \quad (\text{A.174})$$

where

$$\begin{aligned} C'_{n+1} &= \frac{1}{1 - \beta_1 \beta_2} C_{n+1} + \frac{\beta_1}{1 - \beta_1 \beta_2} C_{n+2}, & C'^{\perp}_{n+1} &= C_{n+1}^{\perp} - \beta_2 C_{n+2}^{\perp}, \\ C'_{n+2} &= \frac{1}{1 - \beta_1 \beta_2} C_{n+2} + \frac{\beta_2}{1 - \beta_1 \beta_2} C_{n+1}, & C'^{\perp}_{n+2} &= C_{n+2}^{\perp} - \beta_1 C_{n+1}^{\perp}, \end{aligned} \quad (\text{A.175})$$

and

$$\begin{aligned} \underline{\lambda}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\lambda}_{n+1} &= \xi_A, & \underline{\lambda}_{n+2} &= \xi_B \\ \underline{\tilde{\lambda}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\tilde{\lambda}}_{n+1} &= -\frac{\langle \xi_B | q \rangle}{\langle \xi_B \xi_A \rangle}, & \underline{\tilde{\lambda}}_{n+2} &= -\frac{\langle \xi_A | q \rangle}{\langle \xi_A \xi_B \rangle}, \\ \underline{\tilde{\eta}}_i^+ &= \tilde{\eta}_i^+, & i &= 1, \dots, n, & \underline{\tilde{\eta}}_{n+1}^+ &= 0, & \underline{\tilde{\eta}}_{n+2}^+ &= 0, \\ \underline{\tilde{\eta}}_i^- &= \tilde{\eta}_i^-, & i &= 1, \dots, n, & \underline{\tilde{\eta}}_{n+1}^- &= -\frac{\langle \xi_B | \gamma^- \rangle}{\langle \xi_B \xi_A \rangle}, & \underline{\tilde{\eta}}_{n+2}^- &= -\frac{\langle \xi_A | \gamma^- \rangle}{\langle \xi_A \xi_B \rangle}. \end{aligned} \quad (\text{A.176})$$

The transition from the integration over  $C$ -matrix to integration over  $C'$  matrix is again done similar to the case of off-shell amplitude considered above. This way our form factor is written as

$$\begin{aligned} F_{k,n} &= -\langle \xi_A \xi_B \rangle^2 \int \frac{d^{k \times (n+2)} C'}{\text{Vol}[GL(k)]} \frac{d\beta_1 d\beta_2}{(1 - \beta_1 \beta_2)} \delta^{k \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp} \cdot \underline{\lambda}) \\ &\quad \times \frac{1}{(1 \dots k)' \dots (n+2 \dots k-1)' \left(1 - \beta_1 \frac{(n-k+2 \dots n n+2)'}{(n-k+2 \dots n n+1)'}\right) \left(1 - \beta_2 \frac{(n+1 1 \dots k-1)'}{(n+2 1 \dots k-1)'}\right)} \end{aligned}$$

Finally, taking residues at  $\beta_1 = \frac{(n-k+2 \dots n n+1)'}{(n-k+2 \dots n n+2)'}$  and  $\beta_2 = \frac{(n+2 1 \dots k-1)'}{(n+1 1 \dots k-1)'}$  we reproduce the result of [48]:

$$F_{k,n} = \int \frac{d^{k \times (n+2)} C}{\text{Vol}[GL(k)]} \text{Reg.} \frac{\delta^{k \times 2} (C \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C^{\perp} \cdot \underline{\lambda})}{(1 \dots k)(2 \dots k+1) \dots (n+2 \dots k-1)}, \quad (\text{A.177})$$

where, now

$$\text{Reg.} = \langle \xi_A \xi_B \rangle^2 \frac{Y}{1 - Y}, \quad Y = \frac{(n - k + 2 \dots n n + 1)(n + 2 1 \dots k - 1)}{(n - k + 2 \dots n n + 2)(n + 1 1 \dots k - 1)}. \quad (\text{A.178})$$

In the formula above we assumed a sum over different top-cell forms corresponding to different gluing positions of the minimal form factor. Note, that string correlation function knows about these different top cells by construction.

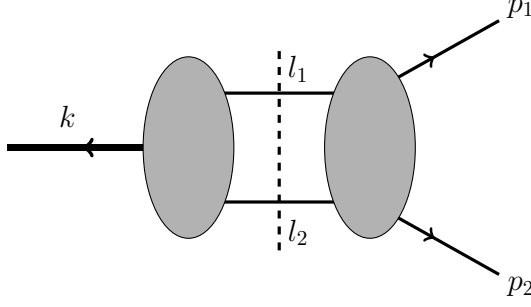


Figure 1: Unitarity cut of  $A_{2,2+1}^{*(1)}$  amplitude in  $k^2 = (p_1 + p_2)^2$  channel.

## B Gluing procedure for the amplitudes integrands

So far we have only considered tree level amplitudes or equivalently string correlation functions of vertex operators on sphere. The key ingredient in the above consideration was a gluing procedure discussed in detail in preceding appendices. The natural question is whether the same gluing operation could be also applied at loop level. To answer this question it is more convenient to consider it at the level of corresponding amplitude integrands. The later are given by combinations of tree level amplitudes and it is likely that gluing procedure will work in this case also. So, we are expecting that the gluing procedure will convert integrands of planar  $\mathcal{N} = 4$  SYM on-shell amplitudes into the corresponding integrands for Wilson line form factors. To see whether it is the case let us consider the simplest possible example of  $k = 2$ ,  $n + 1 = 3$  point one loop amplitude  $A_{2,2+1}^{*(1)}$  and show that the gluing procedure applied to the integrand of  $A_{2,4}^{(1)}$  amplitude will give us the desired expression for the integrand of  $A_{2,2+1}^{*(1)}$  amplitude. But first let's obtain the integrand of  $A_{2,2+1}^{*(1)}$  amplitude. The easiest way to get it is via the use of  $k^2$  channel unitarity cut. Considering the latter (taking residues of the integrand with respect to the poles of  $1/l_1^2$  and  $1/l_2^2$  propagators, see Fig. 1) we have<sup>28</sup>:

$$A_{2,2+1}^{*(1)} \Big|_{k^2 \text{ cut}} = \int d^4\eta_{l_1} d^4\eta_{l_2} A_{2,2+1}^*(g^*, l_1, l_2) A_{2,4}(l_1, l_2, 2, 1) = A_{2,2+1}^*(g^*, l_1, l_2) \frac{\text{Tr}(kp21)}{(pl_2)(l_22)}. \quad (\text{B.179})$$

The  $\text{Tr}$  factor can be transformed into  $k^2(p + p_2)^2 = (p_1 + p_2)^2(p_2 + p)^2$  with the help of momentum conservation  $k + p_1 + p_2 = 0$ , and  $k_T$  decomposition conditions  $(pk) = 0$ . Thus, the expression for  $A_{2,2+1}^{*(1)}(g^*, 1, 2)$  amplitude is given by

$$A_{2,2+1}^{*(1)}(g^*, 1, 2) = A_{2,2+1}^*(g^*, 1, 2) \int d^Dl \frac{(p_1 + p_2)^2(p_2 + p)^2}{l^2(l + p_2)^2(l + p_1 + p_2)^2(pl)}, \quad (\text{B.180})$$

---

<sup>28</sup>The necessary manipulations are similar to the case of  $s$ -channel cut of  $A_{2,4}^{(1)}$  amplitude.

which contains one loop scalar box integral with one of the propagators, namely  $1/l^2$  replaced by its eikonal counterpart  $1/(pl)$ , see Fig. 2.

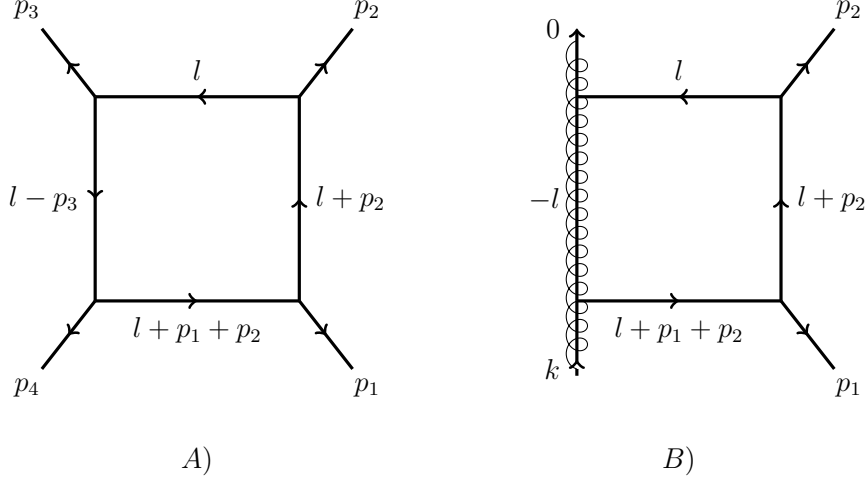


Figure 2: Scalar box integrals contributing to  $A_{2,2+1}^{(1)}$  and  $A_{2,2+1}^{*(1)}$  amplitudes correspondingly. Line with coil denotes eikonal propagator  $1/(pl)$ .

Now let's turn to the integrand of  $A_{2,4}^{(1)}(1, 2, 3, 4)$  amplitude:

$$A_{2,4}^{(1)}(1, 2, 3, 4) = A_{2,4}(1, 2, 3, 4) \int d^D l \frac{(p_1 + p_2)^2 (p_2 + p_3)^2}{l^2 (l + p_2)^2 (l + p_1 + p_2)^2 (l - p_3)^2}. \quad (\text{B.181})$$

It should be noted, that the notion of integrand is uniquely defined only in dual variables. So, to be accurate we should consider the gluing operation in such variables (momentum twistors) also. Here, we will however use helicity spinors in a hope that possible loop momentum rearrangement will not cause any trouble. It turns out that it is indeed the case as we will see in a moment. Acting with  $\hat{A}_{34}$  operator on the  $A_{2,4}^{(1)}(1, 2, 3, 4)$  integrand

$$Int = A_{2,4}(1, 2, 3, 4) \frac{(p_1 + p_2)^2}{l^2 (l + p_2)^2 (l + p_1 + p_2)^2} \frac{(p_2 + p_3)^2}{(l - p_3)^2}. \quad (\text{B.182})$$

and using momentum definitions (6.123) we get

$$\hat{A}_{34}[Int] = A_{2,2+1}^*(g^*, 1, 2) \frac{(p_1 + p_2)^2}{l^2 (l + p_2)^2 (l + p_1 + p_2)^2} \frac{(p_2 + p)^2}{(lp)}, \quad (\text{B.183})$$

which is exactly the integrand expression for  $A_{2,2+1}^{*(1)}$  amplitude.

This example gives us a hope that more accurate and general consideration of gluing procedure at the level of integrands will be also successful and will provide us with the prescription for obtaining  $A_{k,m+n}^{*(l)}$  integrands from the corresponding  $A_{k,m+2n}^{(l)}$  integrands by application of appropriate combinations of  $\hat{A}_{ii+1}$  operators.

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